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Space-time geometry of relativistic particles

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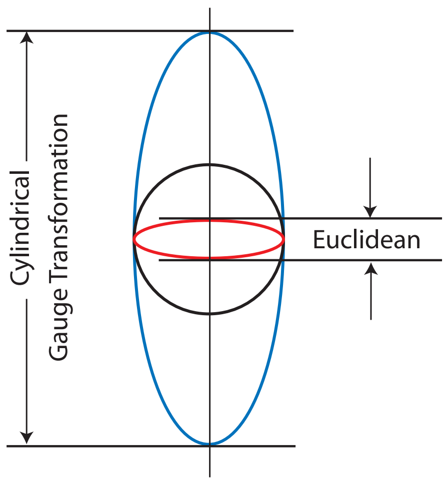
Abstract

A three-dimensional space-time geometry of relativistic particles is constructed within the framework of the little groups of the Poincaré group. Since the little group for a massive particle is the three- dimensional rotation group, its relevant geometry is a sphere. For massless particles and massive particles in the infinite-momentum limit, it is shown that the geometry is that of a cylinder and a two-dimensional plane. The geometry of a massive particle continuously becomes that of a massless particle as the momentum/mass becomes large. The geometry of relativistic extended particles is also considered. It is shown that the cylindrical geometry leads to the concept of gauge transformations, while the two-dimensional Euclidean geometry leads to a deeper understanding of the Lorentz condition. PACS: 03.30.+p, 11.17.+y, 11.30.Cp, 12.40.Aa



1 Introduction

The internal space-time symmetries of relativistic particles are governed by the little groups of the Poincaré group [1, 2]. The internal space-time symmetry group for massive and massless particles are isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean group respectively. We have shown in our previous paper [3] that the internal space-time symmetry of massless particles is dictated by the cylindrical group which is isomorphic to the Euclidean group. The cylindrical axis is parallel to the momentum. For the case of electromagnetic fields satisfying the Lorentz condition, the rotation around the axis corresponds to helicity, while the translation on the surface of the cylinder along the direction of the axis corresponds to a gauge transformation [4].

The purpose of the present paper is to present a more complete geometrical picture of relativistic particles. Since the little groups for massive and massless particles are three-parameter groups [1], it is possible to construct a three-dimensional geometry of internal space-time symmetries for all relativistic particles starting from a sphere for a massive particle at rest. It was observed in Ref. [3] that the three-dimensional rotation group can be contracted either to the two-dimensional Euclidean group or to the cylindrical group [3, 5]. In the present paper, we point out first that both the cylindrical and Euclidean geometries are needed for the little group for massless particles [3, 6].

We shall then show that the Euclidean geometry leads to a deeper understanding of the Lorentz condition applicable to massless particles and to massive particles in the infinite-momentum limit. It is then shown that the cylindrical symmetry is shared by all those particles, even without the requirement of the Lorentz condition. This means that the concept of gauge transformation can be extended to all massless particles or massive particles with infinite momentum.

Also in this paper, we shall discuss relativistic extended particles which are often called hadrons. It is not difficult to visualize the symmetry of an extended particle as the three-dimensional rotation group [7]. However, it is not trivial to construct the geometry of a relativistic extended particle or hadron if it moves with a speed close to that of light. We attack this problem by constructing the generators of the little groups in differential form and the wave functions to which these operators are applicable.

In Sec. 2, we discuss the three-dimensional rotation group and its contractions to the cylindrical and the two-dimensional Euclidean group. It is shown that both of these contractions can be combined into a single representation. In Sec. 3, the generators of the little group are discussed in the light-cone coordinate system. It is shown that these generators are identical with the combined geometry of the cylindrical group and the Euclidean group discussed in Sec. 2.

In Sec. 4, we show that the Lorentz condition is not a prerequisite for the cylindrical symmetry and that the Euclidean symmetry replaces the role of the Lorentz condition. In Sec. 5, the formalism developed in Secs. 2, 3, and 4 is applied to the the space-time geometry of relativistic extended hadrons. It is shown that the relativistic hadron can be described in terms of the parameters of the cylindrical group. Feynman's parton picture is discussed as an illustrative example.

2 Three-dimensional Geometry of the Little Groups

The little groups for massive and massless particles are isomorphic to O(3) and E(2) respectively. It is not difficult to construct the O(3)-like geometry of the little group for a massive particle at rest [1]. The generators L_i of the rotation group satisfy the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk} L_k. \tag{1}$$

Transformations applicable to the coordinate variables x, y, and z are generated by

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

This rotation group is well known.

The Euclidean group E(2) is generated by L_3, P_1 and P_2, with

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

and they satisfy the commutation relations:

$$[P_1, P_2] = 0, \qquad [L_3, P_1] = iP_2, \qquad [L_3, P_2] = -iP_1. \tag{4}$$

The generator L_3 is given in Eq.(2). When applied to the vector space $(x, y, 1)$, P_1 and P_2 generate translations on in the xy plane. The geometry of E(2) is also quite familiar to us.

Let us transpose the above algebra. Then P_1 and P_2 become Q_1 and Q_2, where

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \tag{5}$$

respectively. Together with L_3, these generators satisfy the same set of commutation relations as that for L_3, P_1, and P_2 given in Eq.(4)

$$[Q_1, Q_2] = 0, \qquad [L_3, Q_1] = iQ_2, \qquad [L_3, Q_2] = -iQ_1. \tag{6}$$

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by L_3. The matrices Q_1 and Q_2 generate translations along the direction of axis [3]. We shall call the group generated by these three matrices the *cylindrical group.*

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

$$P_1 = \frac{1}{R} B^{-1} L_2 B, \qquad P_2 = -\frac{1}{R} B^{-1} L_1 B, \tag{7}$$

and

$$Q_1 = -\frac{1}{R} B L_2 B^{-1}, \qquad Q_2 = \frac{1}{R} B L_1 B^{-1}, \tag{8}$$

where

$$B(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}.$$

The vector spaces to which the above generators are applicable are $(x, y, z/R)$ and (x, y, Rz) for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively.

Since $P_1(P_2)$ commutes with $Q_2(Q_1)$, we can consider the following combination of generators.

$$F_1 = P_1 + Q_1, \qquad F_2 = P_2 + Q_2. \tag{9}$$

Then these operators also satisfy the commutation relations:

$$[F_1, F_2] = 0, \qquad [L_3, F_1] = iF_2, \qquad [L_3, F_2] = -iF_1. \tag{10}$$

However, we cannot make this addition using the three-by-three matrices for P_i and Q_i to construct three-by-three matrices for F_1 and F_2, because the vector spaces are different for the P_i and Q_i representations. We can accommodate this difference by creating two different z coordinates, one with a contracted z and the other with an expanded z, namely $(x, y, Rz, z/R)$. Then the generators become

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

Then F_1 and F_2 will take the form:

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The rotation generator L_3 takes the form

$$L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

These four-by-four matrices satisfy the E(2)-like commutation relations of Eq.(10).

Next, let us consider the transformation matrix generated by the above matrices. It is easy to visualize the transformations generated by P_i and Q_i. It would be easy to visualize the transformation generated by F_1 and F_2, if P_i commuted with Q_i. However, P_i and Q_i do not commute with each other, and the transformation matrix takes a somewhat complicated form:

$$\exp\{-i(\xi F_1 + \eta F_2)\} = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

If we make a similarity transformation on the above form using the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \tag{16}$$

which performs a 45-degree rotation of the third and fourth coordinates, then $\exp\{-i(\xi F_1 + \eta F_2)\}$ of Eq.(15) becomes

$$\begin{pmatrix} 1 & 0 & -\xi/\sqrt{2} & \xi/\sqrt{2} \\ 0 & 1 & -\eta/\sqrt{2} & \eta/\sqrt{2} \\ \xi/\sqrt{2} & \eta/\sqrt{2} & 1 - (\xi^2 + \eta^2)/4 & (\xi^2 + \eta^2)/4 \\ \xi/\sqrt{2} & \eta/\sqrt{2} & -(\xi^2 + \eta^2)/4 & 1 + (\xi^2 + \eta^2)/4 \end{pmatrix}. \tag{17}$$

This form is readily available in the literature [1, 4] as the translation-like transformation matrix for the little group for massless particles. In this section, we have given a geometrical interpretation to this matrix.

3 Little Groups in the Light-cone Coordinate System

Let us now study the group of Lorentz transformations using the light-cone coordinate system. If the space-time coordinate is specified by (x, y, z, t), then the light-cone coordinate variables are (x, y, u, v) for a particle moving along the z direction, where

$$u = (z + t)/\sqrt{2}, \qquad v = (z - t)/\sqrt{2}. \tag{18}$$

The transformation from the conventional space-time coordinate to the above system is achieved through a similarity transformation of Eq.(16).

In the light-cone coordinate system, the generators of Lorentz transformations are

$$\begin{aligned}
J_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad K_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
J_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i & -i \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
J_3 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.
\end{aligned} \tag{19}$$

where J_1, J_2, and J_3 are the rotation generators, and K_1, K_2, and K_3 are the generators of boosts along the three orthogonal directions.

If a massive particle is at rest, its little group is generated by J_1, J_2 and J_3. For a massless particle moving along the z direction, the little group is generated by N_1, N_2 and J_3, where

$$N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1, \tag{20}$$

which can be written in the matrix form as

$$N_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_1, \qquad [N_1, N_2] = 0. \tag{22}$$

Let us go back to F_1 and F_2 of Eq.(13). Indeed, they are proportional to N_1 and N_2 respectively:

$$N_1 = \frac{1}{\sqrt{2}}F_1, \qquad N_2 = \frac{1}{\sqrt{2}}F_2. \tag{23}$$

Since F_1 and F_2 are somewhat simpler than N_1 and N_2, and since the commutation relations of Eq.(22) are invariant under multiplication of N_1 and N_2 by constant factors, we shall hereafter use F_1 and F_2 for N_1 and N_2.

In the light-cone coordinate system, the boost matrix takes the form

$$B(R) = \exp\left(-i\rho K_3\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}, \tag{24}$$

with $\rho = \ln(R)$, and $R = \sqrt{(1+\beta)/(1-\beta)}$, where β is the velocity parameter of the particle. The boost is along the z direction. Under this transformation, x and y coordinates are invariant, and the light-cone variables u and v are transformed as

$$u' = Ru, \qquad v' = v/R. \tag{25}$$

If we boost J_2 and J_1 and multiply them by $\sqrt{2}/R$, as

$$W_1(R) = -\frac{\sqrt{2}}{R}BJ_2B^{-1} = \begin{pmatrix} 0 & 0 & -i/R^2 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i/R^2 & 0 & 0 & 0 \end{pmatrix},$$

$$W_2(R) = \frac{\sqrt{2}}{R}BJ_1B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i/R^2 & i \\ 0 & i & 0 & 0 \\ 0 & i/R^2 & 0 & 0 \end{pmatrix}, \tag{26}$$

then $W_1(R)$ and $W_2(R)$ become F_1 and F_2 of Eq.(13) respectively in the large-R limit.

The algebra given in this section is identical with that of Sec. 2 based on the three-dimensional geometry of a sphere going through a contraction/expansion of the z axis.

Therefore, it is possible to give a concrete geometrical picture to the little groups of the Poincaré group governing the internal space-time symmetries of relativistic particles.

The most general form of the transformation matrix is

$$D(\xi, \eta, \alpha) = D(\xi, \eta, 0) D(0, 0, \alpha), \tag{27}$$

with

$$D(\xi, \eta, 0) = \exp\{-i(\xi F_1 + \eta F_2)\}, \qquad D(0, 0, \alpha) = \exp(-i\alpha J_3). \tag{28}$$

The matrix $D(0, 0, \alpha)$ represents a rotation around the z axis. In the light-cone coordinate system, $D(\xi, \eta, 0)$ takes the form of Eq.(15). It is then possible to decompose it into

$$D(\xi, \eta, 0) = C(\xi, \eta) E(\xi, \eta) S(\xi, \eta), \tag{29}$$

where

$$\begin{aligned}
C(\xi, \eta) &= \exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
E(\xi, \eta) &= \exp(-i\xi P_1 - i\eta P_2) = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
S(\xi, \eta) &= I + \frac{1}{2}\left[C(\xi, \eta), E(\xi, \eta)\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{30}$$

The matrix $C(\xi, \eta)$ performs a cylindrical transformation on the first, second and third components, while $E(\xi, \eta)$ is for a Euclidean transformation on the first, second and fourth components. The matrix $S(\xi, \eta)$ performs a translation along the third axis and commutes with both $C(\xi, \eta)$ and $E(\xi, \eta)$. Both $E(\xi, \eta)$ and $S(\xi, \eta)$ become identity matrices when applied to four-vectors with vanishing fourth component. We shall study physical implications of these properties in the following section.

4 Cylindrical Group and Gauge Transformations

Let us consider a particle represented by a four-vector:

$$A^{\mu}(x) = A^{\mu} e^{i(kz - \omega t)}, \tag{31}$$

where $A^{\mu} = (A_1, A_2, A_3, A_0)$. This is not a massless particle. In the light-cone coordinate system,

$$A^{\mu} = (A_1, A_2, A_u, A_v) \,, \tag{32}$$

where $A_u = (A_3 + A_0)/\sqrt{2}$, and $A_v = (A_3 - A_0)/\sqrt{2}$. If it is boosted by the matrix of Eq.(24), then

$$A'^{\mu} = (A_1, A_2, RA_u, A_v/R) \,. \tag{33}$$

Thus the fourth component will vanish in the large-R limit, while the third component becomes large.

The momentum-energy four-vector in the light-cone coordinate system is

$$P^{\mu} = \left(0, 0, (k+\omega)/\sqrt{2}, (k-\omega)/\sqrt{2}\right) , \tag{34}$$

which in the rest frame becomes

$$P^{\mu} = \left(0, 0, m/\sqrt{2}, -m/\sqrt{2}\right) , \tag{35}$$

where m is the mass. If we boost this four-momentum using the matrix of Eq.(24), then

$$P'^{\mu} = \left(0, 0, Rm/\sqrt{2}, -m/\sqrt{2}R\right) . \tag{36}$$

Here again, the fourth component vanishes for large values of R, while the third component becomes large.

Let us go back to $W_1(R)$ and $W_2(R)$ of Eq.(26). If $W_1(R)$ is applied to the four-vector A'^{μ}, the result is

$$i\left((A_u - A_v)/R, 0, A_1, -A_1/R^2\right) , \tag{37}$$

which becomes $(0, 0, -iA_1, 0)$ for large values of R. When $W_2(R)$ is applied, the result is $(0, 0, -iA_2, 0)$. Thus, the i/R^2 factors in $W_1(R)$ and $W_2(R)$ can be dropped in the large-R limit. We can thus safely apply the transformation matrix generated by F_1 and F_2 of Eq.(13).

Since the fourth component of the vector vanishes or becomes vanishingly small, the application of $S(\xi, \eta)$ of Eq.(30) on A'^{μ} and P'^{μ} will be produce no effects in the large$-R$ limit. The same is true for $E(\xi, \eta)$ of Eq.(30). Thus, among the three factors of the transformation matrix, only the matrix $C(\xi, \eta)$ given in Eq.(30) will produce a nontrivial effect. This is the cylindrical transformation [3].

During the limiting process, the three-dimensional geometry consisting of the x, y, and v coordinates describes a pancake-like compression of the sphere in which the v coordinate shrinks to zero. Because of this contraction of the v coordinate, the Euclidean component of the little group disappears. This is the content of the Lorentz condition for massive

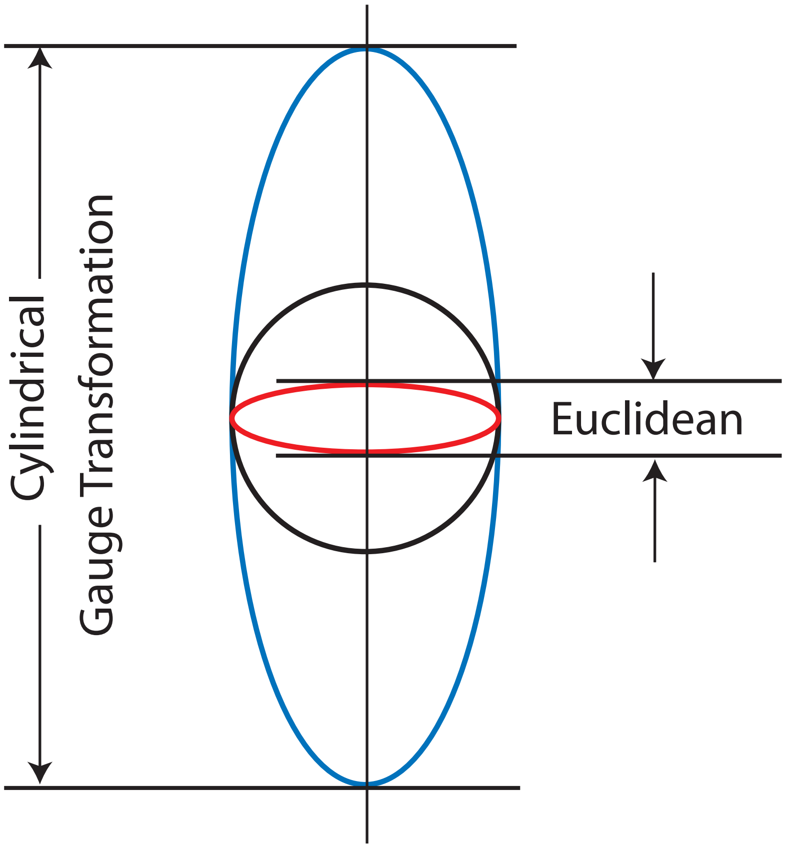


Figure 1: Cylindrical and Euclidean deformations of the sphere. It is possible to contract the z axis by dividing it by R. This contraction of the z axis leads to the contraction of $O(3)$ to the two-dimensional Euclidean group. If the z axis is multiplied by R, then it becomes expanded. This expansion of the z axis leads to the contraction of $O(3)$ to the cylindrical group. The expanding and contracting z axis are treated as different coordinates, and are called the u and v coordinates respectively in Secs. 3, 4, and 5.

particles in the infinite-momentum limit. The three-dimensional geometry of the x, y and u coordinates corresponds to the expanding z coordinate, resulting in the cylindrical symmetry. See Fig. 1.

Let us see the effect of $C(\xi, \eta)$ on the four-vector of Eq.(33). If we apply $C(\xi, \eta)$ to the four-vector, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ RA_u \\ A_v/R \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ RA_u + \xi A_1 + \eta A_2 \\ A_v/R \end{pmatrix}. \tag{38}$$

This transformation produces only additions to the third component, which are gauge transformations in the case of electromagnetic four-potential.

The above transformation is not unlike the $D(\xi, \eta)$ transformation applied to the four-vector satisfying the Lorentz condition $A_v = 0$:

$$\begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ RA_u \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{39}$$

which is equal to

$$\begin{pmatrix} A_1 \\ A_2 \\ RA_u \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ RA_u + \xi A_1 + \eta A_2 \\ 0 \end{pmatrix}. \tag{40}$$

As we noted at the end of Sec. 3, the Lorentz condition eliminates the Euclidean component in the $D(\xi, \eta, 0)$ matrix. It is remarkable that Eq.(39) is strikingly similar to Eq.(38). The cylindrical transformation is quite independent of the fourth component in both cases, and it produces the same result for the first three components. Thus the elimination of the Euclidean component which led to Eq.(38) can thus be regarded as an extension of the Lorentz condition to all four-vectors.

5 Little Groups for Relativistic Extended Particles

We are now ready to discuss the symmetry property discussed in Sec. 3 for relativistic extended particles or hadrons. Let us consider a hadron consisting of two quarks bound together by an attractive force such as the harmonic oscillator force. We use four-vectors x_a

and x_b to specify space-time positions of the two quarks. Then it is more convenient to use the following variables [8].

$$X = (x_a + x_b)/2, \qquad x = (x_a - x_b)/2\sqrt{2}. \tag{41}$$

The four-vector X specifies where the hadron is located in space- time, while the variable x measures the space-time separation between the quarks.

In the light-cone coordinate system, the generators of rotations applicable to functions localized in the four-dimensional space-time of x are

$$\begin{aligned}
J_1 &= -\frac{i}{\sqrt{2}}\left\{y\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) - (u+v)\frac{\partial}{\partial y}\right\}, \\
J_2 &= -\frac{i}{\sqrt{2}}\left\{x\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) - (u+v)\frac{\partial}{\partial x}\right\}, \\
J_3 &= -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right).
\end{aligned} \tag{42}$$

The boost generators are

$$\begin{aligned}
K_1 &= -\frac{i}{\sqrt{2}}\left\{x\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) + (u-v)\frac{\partial}{\partial y}\right\}, \\
K_2 &= -\frac{i}{\sqrt{2}}\left\{y\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) + (u-v)\frac{\partial}{\partial x}\right\}, \\
K_3 &= -i\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right).
\end{aligned} \tag{43}$$

These generators do not contain the hadronic coordinate variable X, as transformations of the little group do not change the hadronic momentum. The boost operator along the z direction is

$$B(R) = \exp\left\{-\rho\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)\right\}. \tag{44}$$

If this boost is applied to J_2 and J_1, as in the case of Eq.(26),

$$\begin{aligned}
W_1(R) &= -i\left\{x\frac{\partial}{\partial u} - v\frac{\partial}{\partial x} - \left(\frac{1}{R}\right)^2\left(u\frac{\partial}{\partial x} - x\frac{\partial}{\partial v}\right)\right\}, \\
W_2(R) &= -i\left\{y\frac{\partial}{\partial u} - v\frac{\partial}{\partial y} - \left(\frac{1}{R}\right)^2\left(u\frac{\partial}{\partial y} - y\frac{\partial}{\partial v}\right)\right\}.
\end{aligned} \tag{45}$$

In the limit of large R, W_1 and W_2 become F_1 and F_2 respectively [9]:

$$F_1 = -i\left(x\frac{\partial}{\partial u} - v\frac{\partial}{\partial x}\right), \qquad F_2 = -i\left(y\frac{\partial}{\partial u} - v\frac{\partial}{\partial y}\right). \tag{46}$$

The transformation operator is now

$$D(\xi, \eta, 0) = \exp\left\{-i(\xi x + \eta y)\frac{\partial}{\partial u} - iv\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\right\}, \tag{47}$$

which can be decomposed into

$$\begin{aligned} = \quad D(\xi, \eta, 0) = \exp\left\{-i(\xi x + \eta y)\frac{\partial}{\partial u}\right\}\exp\left\{-iv\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\right\} \\ \times\exp\left\{-i\frac{v}{2}\left(\xi^2 + \eta^2\right)\frac{\partial}{\partial u}\right\}, \end{aligned} \tag{48}$$

as in the case of Eq.(29).

We are applying this operator on functions localized in the four-dimensional space-time. As an illustration, let us consider Dirac's Gaussian form [10]:

$$\psi(x) = \frac{1}{\pi}\exp\left\{-\left(\frac{x^2 + y^2 + z^2 + t^2}{2}\right)\right\}. \tag{49}$$

This form is not invariant under Lorentz boosts, but undergoes a Lorentz deformation when the system is boosted [9, 11]. If it is boosted along the z direction, the x and y coordinates are not affected. We can therefore delete these transverse variables, and concentrate on the Lorentz deformation property of

$$\psi(z, t) = \frac{1}{\sqrt{\pi}}\exp\left\{-\left(\frac{u^2 + v^2}{2}\right)\right\}. \tag{50}$$

in the uv plane. The light-cone variables u and v are defined in Eq.(18), and their Lorentz-transformation property is given in Eq.(25). If this function is Lorentz boosted along the z axis,

$$\psi_\beta(z, t) = \frac{1}{\sqrt{\pi}}\exp\left\{-\left(\frac{(u/R)^2 + (Rv)^2}{2}\right)\right\}. \tag{51}$$

The width of this function along the u axis increases as R becomes large, while the distribution along the v axis becomes narrow, as is described in Fig. 2.

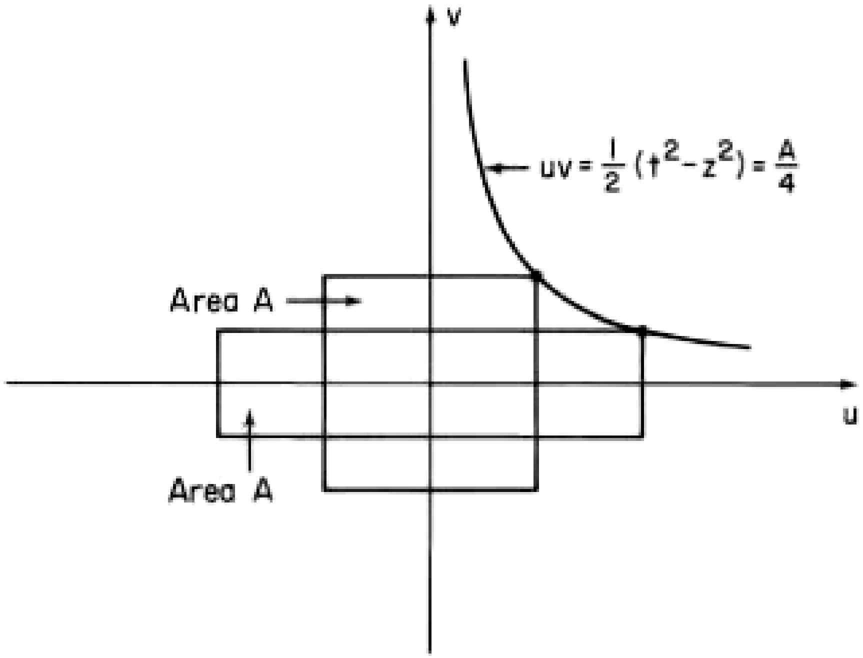


Figure 2: Lorentz deformation in the uv plane. As the velocity parameter increases, the distribution along the u axis becomes expanded while the v axis becomes contracted, in such a way that the area remains constant. In the infinite-momentum limit, the v distribution becomes like that of $\delta(v)$, while the distribution along the u axis becomes wide-spread. The translation along the u axis becomes a gauge transformation.

This function illustrates the Lorentz-deformation property of functions localized in the uv plane. The width of the v distribution decreases as 1/R. When the v distribution is very narrow, we can consider the transformation in the subspace where $v = 0$. Then the factors

$$\exp\left\{-iv\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\right\} \tag{52}$$

and

$$\exp\left\{-i\frac{v}{2}\left(\xi^2 + \eta^2\right)\frac{\partial}{\partial u}\right\}, \tag{53}$$

in Eq.(**??**) for $D(\xi, \eta, 0)$ can be dropped. As a consequence,

$$D(\xi, \eta, 0) = \exp\left\{-i(\xi x + \eta y)\frac{\partial}{\partial u}\right\}. \tag{54}$$

This means that the terms $v\frac{\partial}{\partial x}$ and $v\frac{\partial}{\partial y}$ in Eq.(46) can be dropped, and F_1 and F_2 can be written

$$F_1 = -ix\frac{\partial}{\partial u}, \qquad F_2 = -iy\frac{\partial}{\partial u}. \tag{55}$$

These operators generate translations along the u axis. These operators, together with the rotation generator J_3 of Eq.(42), are the generators of the cylindrical group. The differential operators F_1 and F_2 are now the generators of gauge transformations applicable to functions with a narrow distribution in v [9].

Here again, a complete description of the little group for massive particles in the infinite-momentum limit require both the cylindrical and Euclidean components. The Euclidean component can be deleted in the infinite-momentum limit or in the $v = 0$ subspace. As we observed at the end of Sec. 4, this is the Lorentz condition applicable to massive particles in the infinite-momentum limit.

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