Journal of Modern Optics
2009, 1–6, iFirst

Optical activities as computing resources for space–time symmetries

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(*Received 1 December 2008; final version received 2 February 2009*)

It is known that optical activities can perform rotations. It is shown that the rotation, if modulated by attenuations, can perform symmetry operations of Wigner's little group which dictates the internal space–time symmetries of elementary particles.

Keywords: optical activities; Lorentz group; internal space–time symmetries

1. Introduction

Polarization optics serves as analog computers for the geometry of ellipsometry [1] and the Poincaré sphere [2]. They also can perform algebras of two-by-two and four-by-four matrices known as Jones and Mueller matrices, respectively [1,3]. It was established recently that these matrices correspond to the two-by-two and four-by-four representations of the Lorentz group which serves as the mathematical framework for Einstein's special relativity [4–6].

Optical activities are known to speak the language of rotations. In the real world, all optical rays go through attenuations. If the attenuation is axially symmetric, it does not raise additional mathematical problems.

In this paper, let us assume that the optical ray propagates along the z direction, and that the polarization rotates on the xy plane. If the attenuation along the x direction is different from that along the y direction, the combined effect of this asymmetric attenuation and the rotation around the z axis can perform an interesting mathematical operation.

It is a simple matter to construct a rotation matrix for a given value of propagation distance Z. So is the matrix for the asymmetric attenuation. However, the problem becomes nontrivial when these two effects are combined at a microscopic scale with a small value of z, and this combined effect is repeated to make up the finite value of z.

It is shown in this paper that the resulting mathematics not only allows one to make analytical calculations of the optical activities with asymmetric attenuation effects, but also provides a computational instrument for Wigner's little group which dictates the internal space–time symmetries of elementary particles.

In 1939, Wigner noted a particle can have internal variables in addition to its energy and momentum [7]. For instance, an electron can have its spin degrees of freedom, in addition to its momentum and energy. Photons can have helicity and gauge degrees of freedom. Wigner formulated this symmetry problem by introducing a three-parameter subgroup of the Lorentz group which preserves the four-momentum of a given particle. This subgroup is called Wigner's little group in the literature.

For a massive particle, the little group is a Lorentz-boosted rotation group. For a massless particle, it is like (locally isomorphic to) the two-dimensional Euclidean group. For a tachyon with a space-like four-momentum, the little group is a Lorentz-boosted boost matrix, where the two boosts are made along perpendicular directions [7,8].

It is now possible to understand optical activities in terms of Wigner's little group. Conversely, the optical activity can serve as an analog computer for internal space–time symmetries of elementary particles.

In Section 2, we formulate the problem in terms of two-by-two matrices applicable to the Jones vector. It is not difficult to write matrices performing rotations and attenuations separately. In Section 3, we compute the transformation matrix if those two operations are performed at a microscopic scale, and are accumulated to a macroscopic scale. In Section 4, it is shown that the transformation matrices correspond to those of Wigner's little group which dictates internal space–time symmetries of elementary particles.

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ISSN 0950–0340 print/ISSN 1362–3044 online

DOI: 10.1080/09500340902795984
http://www.informaworld.com

2. Formulation of the problem

Let us start with a light wave taking the form

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A\exp\{\mathrm{i}(kz-\omega t+\phi_1)\} \\ B\exp\{\mathrm{i}(kz-\omega t+\phi_2)\} \end{pmatrix}. \tag{1}$$

This ray can go through rotations around the z axis. It can also go through xy asymmetric phase shifts and attenuations. The mathematics of these aspects is known as the Jones matrix formalism. If there are decoherence effects between the x and y components, the mathematics can be extended to the four-by-four Mueller matrix formalism [2].

It has been recently established that the Jones and Muller formalisms constitute the two-by-two and four-by-four representations of the Lorentz group [4,5]. In the two-by-two case, the transformation matrix is generated by three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

plus three squeeze generators

$$\tau_1 = \mathrm{i}\sigma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tau_2 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\tau_3 = \mathrm{i}\sigma_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{3}$$

There are therefore six generators. They form a closed set of commutation relations. In mathematical terms, they form the Lie algebra for the SL(2, c) group which is locally isomorphic to the six-parameter Lorentz group.

The three matrices of Equation (2) generate the rotation subgroup of the Lorentz group. For possible computer applications, we are interested in the subgroup which produces real transformation matrices. Imaginary numbers are not convenient for computer mathematics.

Among the above six generators given in Equations (2) and (3), σ_2, τ_3, and τ_1 are pure imaginary and can generate real transformation matrices. They satisfy the following closed set of commutation relations.

$$[\sigma_2, \tau_3] = 2\mathrm{i}\tau_1, \quad [\tau_3, \tau_1] = -2\mathrm{i}\sigma_2, \quad [\tau_1, \sigma_2] = 2\mathrm{i}\tau_3. \tag{4}$$

This group generated by these three matrices is called the $Sp(2)$ group and is applicable to many optical instruments and optical processes, either directly or indirectly through its isomorphism with the $S(1,1)$ group for squeezed states.

Within this framework, we are dealing with rotation matrices of the form

$$R(\theta) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{5}$$

and squeeze matrices of the form

$$S(\alpha) = \begin{pmatrix} \exp(\beta) & 0 \\ 0 & \exp(-\beta) \end{pmatrix}, \tag{6}$$

and their multiplications. All the matrices in this representation are real.

Indeed, optical activities can be described by these real matrices. The polarization goes through the rotation

$$R(\gamma z) = \begin{pmatrix} \cos(\gamma z) & -\sin(\gamma z) \\ \sin(\gamma z) & \cos(\gamma z) \end{pmatrix}, \tag{7}$$

as the ray propagates along the z direction. This matrix is applicable to the x and y components of the polarization, and the rotation angle increases as z increases.

The optical ray is expected to be attenuated due to absorption by the medium. The attenuation coefficient in one transverse direction could be different from the coefficient along the other direction. Thus, if the rate of attenuation along the x direction is different from that along the y axis, this asymmetric attenuation can be described by

$$\begin{pmatrix} \exp(-\mu_1 z) & 0 \\ 0 & \exp(-\mu_2 z) \end{pmatrix} = \exp(-\lambda z)\begin{pmatrix} \exp(\mu z) & 0 \\ 0 & \exp(-\mu z) \end{pmatrix}, \tag{8}$$

with

$$\lambda = \frac{\mu_2 + \mu_1}{2}, \qquad \mu = \frac{\mu_2 - \mu_1}{2}. \tag{9}$$

The exponential factor $\exp(-\lambda z)$ is for the overall attenuation, and the matrix

$$\begin{pmatrix} \exp(\mu z) & 0 \\ 0 & \exp(-\mu z) \end{pmatrix}, \tag{10}$$

performs a squeeze transformation. This matrix expands the x component of the polarization, while contracting the y component. We shall call this the squeeze along the x direction.

The squeeze does not have to be along the x direction. It can be in the direction which makes an angle θ with the x axis. The squeeze matrix then becomes

$$S(\theta, \mu z) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \exp(\mu z) & 0 \\ 0 & \exp(-\mu z) \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{11}$$

which can be compressed to one matrix

$$\begin{pmatrix} \cosh(\mu z)+\sin(2\theta)\sinh(\mu z) & \sin(2\theta)\sinh(\mu z) \\ \sin(2\theta)\sinh(\mu z) & \cosh(\mu z)-\sin(2\theta)\sinh(\mu z) \end{pmatrix}. \tag{12}$$

If $\theta = 45^\circ$, this matrix becomes

$$S(\pi/4, \mu z) = \begin{pmatrix} \cosh(\mu z) & \sinh(\mu z) \\ \sinh(\mu z) & \cosh(\mu z) \end{pmatrix}. \tag{13}$$

We shall work with this form of squeeze matrix in the following discussion, and use the notation $S(\mu z)$ without angle for the above expression. Thus, if the squeeze is made along the x axis, the squeeze matrix is

$$S(0, \mu z) = R(-\pi/4, \mu z)S(\mu z)R(\pi/4, \mu z). \tag{14}$$

If this squeeze is followed for the rotation of Equation (7), the net effect is

$$e^{-\lambda z}\begin{pmatrix} \cos(\gamma z) & -\sin(\gamma z) \\ \sin(\gamma z) & \cos(\gamma z) \end{pmatrix}\begin{pmatrix} \cosh(\mu z) & \sinh(\mu z) \\ \sinh(\mu z) & \cosh(\mu z) \end{pmatrix}, \tag{15}$$

where z is in a macroscopic scale, perhaps measured in centimeters. However, this is not an accurate description of the optical process.

This happens in a microscopic scale of z/N, and becomes accumulated into the macroscopic scale of z after N repetitions, where N is a very large number. We are thus led to the transformation matrix of the form

$$M(\gamma, \mu, z) = [\exp(-\lambda z/N)S(\mu z/N)R(\gamma z/N)]^N. \tag{16}$$

In the limit of large N, this quantity becomes

$$\exp(-\lambda z)\left[\begin{pmatrix} 1 & \mu z/N \\ \mu z/N & 0 \end{pmatrix}\begin{pmatrix} 1 & -\gamma z/N \\ \gamma z/N & 1 \end{pmatrix}\right]^N. \tag{17}$$

Since $\gamma z/N$ and $\mu z/N$ are very small,

$$M(\gamma, \mu, z) = \exp(-\lambda z) \times\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -(\gamma-\mu) \\ (\gamma+\mu) & 0 \end{pmatrix}\frac{z}{N}\right]^N. \tag{18}$$

For large N, we can write this matrix as

$$M(\gamma, \mu, z) = \exp(-\lambda z)\exp(Gz), \tag{19}$$

with

$$G = \begin{pmatrix} 0 & -(\gamma-\mu) \\ (\gamma+\mu) & 0 \end{pmatrix}. \tag{20}$$

The remaining problem is to calculate the exponential form $\exp(Gz)$ by making a Taylor expansion. We thus need to compute G^N. This is a trivial problem if G is diagonal or can be diagonalized by a similarity transformation of a diagonal matrix. The problem arises because this is not always the case.

3. Computation of the transformation matrix

We are interested in computing the exponential form of Equation (18). If γ in Equation (20) is greater than μ, the off-diagonal elements have opposite signs, and we can write G as

$$G = k\begin{pmatrix} 0 & -\exp(2\eta) \\ \exp(-2\eta) & 0 \end{pmatrix}, \tag{21}$$

with

$$k = (\gamma^2-\mu^2)^{1/2}, \quad \exp(2\eta) = \left(\frac{\gamma+\mu}{\gamma-\mu}\right)^{1/2}, \tag{22}$$

or conversely

$$\gamma = k\cosh(2\eta), \quad \mu = k\sinh(2\eta). \tag{23}$$

If μ is greater than γ, the off-diagonal elements have the same sign. We can then write G as

$$G = k\begin{pmatrix} 0 & \exp(2\eta) \\ \exp(-2\eta) & 0 \end{pmatrix}, \tag{24}$$

with

$$k = (\mu^2-\gamma^2)^{1/2}, \quad \exp(2\eta) = \left(\frac{\mu+\gamma}{\mu-\gamma}\right)^{1/2}, \tag{25}$$

or

$$\gamma = k\sinh(2\eta), \quad \mu = k\cosh(2\eta). \tag{26}$$

If $\gamma = \mu$, the upper-right element of the G matrix has to vanish, and it becomes

$$\begin{pmatrix} 0 & 0 \\ 2\gamma & 0 \end{pmatrix}. \tag{27}$$

As μ becomes larger from $\mu < \gamma$ to $\mu > \gamma$, the G matrix has to go through this triangular form.

We are now ready to compute the exponential form

$$\exp(zG). \tag{28}$$

The problem is whether it is possible to obtain an analytical expression of the above quantity. The usual procedure is to write a Taylor expansion. For this purpose, we need to calculate G^N. We can manage this calculation when $N = 2$. However, for an arbitrary large integer N, it is not a trivial problem.

This is exactly the problem we would like to address in this section.

If γ is greater than μ, we write G of Equation (21) as

$$G = k\begin{pmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \exp(-\eta) & 0 \\ 0 & \exp(\eta) \end{pmatrix}, \tag{29}$$

with η being given in Equation (22). This is a similarity transformation of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{30}$$

with respect to a squeeze matrix

$$B(\eta) = \begin{pmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{pmatrix}. \tag{31}$$

The role of this squeeze matrix is quite different from that of Equation (13). It does not depend on z.

Let us go back to the G matrix. We can write G^N as

$$k^N\begin{pmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^N\begin{pmatrix} \exp(-\eta) & 0 \\ 0 & \exp(\eta) \end{pmatrix}, \tag{32}$$

and

$$\exp\left[k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos(kz) & -\sin(kz) \\ \sin(kz) & \cos(kz) \end{pmatrix}. \tag{33}$$

Thus, the exponential form $\exp(Gz)$ of Equation (28) becomes

$$\begin{pmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{pmatrix}\begin{pmatrix} \cos(kz) & -\sin(kz) \\ \sin(kz) & \cos(kz) \end{pmatrix}\begin{pmatrix} \exp(-\eta) & 0 \\ 0 & \exp(\eta) \end{pmatrix}, \tag{34}$$

and the transformation matrix of Equation (16) takes the form

$$M(\gamma, \mu, z) = \exp(-\lambda z) \times \begin{pmatrix} \cos(kz) & -\exp(2\eta)\sin(kz) \\ \exp(-2\eta)\sin(kz) & \cos(kz) \end{pmatrix}, \tag{35}$$

with k and η being given in Equation (22).

If μ is greater than γ, the off-diagonal elements of Equation (20) have the same sign, but we can go through a similar calculation. The result is

$$M(\gamma, \mu, z) = \exp(-\lambda z) \times \begin{pmatrix} \cosh(kz) & \exp(2\eta)\sinh(kz) \\ \exp(-2\eta)\sinh(kz) & \cosh(kz) \end{pmatrix}, \tag{36}$$

with k and η being given in Equation (25).

If γ and μ are equal, the G matrix becomes

$$G = \begin{pmatrix} 0 & 0 \\ 2\gamma & 0 \end{pmatrix}, \tag{37}$$

with the property

$$G^2 = \begin{pmatrix} 0 & 0 \\ 2\gamma & 0 \end{pmatrix}^2 = 0, \tag{38}$$

and the transformation matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 2\gamma z & 1 \end{pmatrix}. \tag{39}$$

Let us go back to the case with $\gamma > \mu$. We can then gradually increase the parameter μ to a value greater than γ, which means from Equation (22) to Equation (25). This involves a singularity in the expression of $\exp(2\eta)$ in these equations. This is a complicated mathematical issue [9], but we can avoid the problem using the variables μ and γ.

4. Space–time symmetries spoken by optical activities

As mentioned in Section 1, the Lorentz group provides the basic mathematical framework for polarization optics. The Lorentz group was used earlier by Einstein to formulate his special theory of relativity. In 1905, Einstein considered only point particles. After the formulation of quantum mechanics in 1927, it was found that particles can have internal space–time structures.

If a given particle has internal space–time symmetries, such as electron spin and quark distribution inside a hadron, we have to rely on Wigner's little groups [7]. If the particle is massive, there is a Lorentz frame in which the particle is at rest. In this frame, the four-momentum remains invariant under rotations. However, its spin can change its orientation. Wigner's little group in this case is like (locally isomorphic to) the three-dimensional rotation group. We call this the $O(3)$-like little group for massive particles.

We do not observe particles with space-like momentum or moving faster than light, but they play important roles in physical theories. We need those space-like particles in Feynman diagrams. For a particle of this type, there is the Lorentz frame where the momentum does not have its time-like component. It has its space component along a given direction. This four-momentum is invariant under Lorentz boots along the two perpendicular directions. The subgroup in this case is the Lorentz group applicable to two space coordinates and one time variable. We call this the $O(2, 1)$-like subgroup.

If the particle is massless, like photons, there are no Lorentz frames in which it is at rest or with a zero time-like component. For this case, Wigner in 1939 observed that there is a three-parameter subgroup of the Lorentz group which leaves the four-momentum invariant, and that its algebraic property is the same as that of the two-dimensional Euclidean group. We call this the $E(2)$-like little group for massless particles.

Let us now translate the two-by-two matrices given in Sections 2 and 3 into the language of four-by-four Lorentz transformation matrices applicable to the Minkowski space of (z, y, z, t). In this convention, the momentum-energy four-vector is (p_x, p_y, p_z, E). If the particle moves along the z direction, this four-vector becomes

$$\left(0, 0, p, (p^2 + m^2)^{1/2}\right), \tag{40}$$

in the unit system where $c = 1$, where m is the particle mass. We can obtain this four-vector by boosting a particle at rest with the four-momentum

$$(0, 0, 0, m), \tag{41}$$

with the boost matrix

$$B(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(2\eta) & \sinh(2\eta) \\ 0 & 0 & \sinh(2\eta) & \cosh(2\eta) \end{pmatrix}, \tag{42}$$

where

$$\tanh(2\eta) = \frac{p}{(p^2 + m^2)^{1/2}}. \tag{43}$$

It is known that this boost matrix corresponds to the squeeze matrix $B(\eta)$ of Equation (31) [5].

Now the four-momentum of Equation (41) is invariant under the rotation matrix

$$R(\gamma z) = \begin{pmatrix} \cos(2\gamma z) & 0 & \sin(2\gamma z) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(2\gamma z) & 0 & \cos(2\gamma z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{44}$$

Thus, the matrix

$$B(\eta) R(\gamma z) B(-\eta) \tag{45}$$

leaves the four-momentum of Equation (40) invariant. While this matrix performs a rotation around the y axis in the particle's rest frame, we can also rotate this four-momentum around the z axis without changing it. This is what Wigner's little group is about for the particle with mass m.

Although the matrix of Equation (45) does not change the momentum, it rotates the spin direction of the particle in its rest frame. This is why the little group is not a trivial mathematical device.

It is known that the rotation matrix of Equation (44) corresponds to the rotation matrix of Equation (7) [5]. Thus, the two-by-two rotation matrix of Equation (7), together with the squeeze matrix of Equation (31), generates the little group for particles with non-zero mass.

If the particle has a space-like momentum, we can start with the four-momentum

$$(0, 0, p, E), \tag{46}$$

where E is smaller than p, which can be brought to the Lorentz frame where the four-vector becomes

$$(0, 0, p, 0). \tag{47}$$

The boost matrix takes the same form as Equation (42), with

$$\tanh(2\eta) = \frac{E}{p}. \tag{48}$$

The four-momentum of Equation (47) is invariant under the boost

$$S(\mu z) = \begin{pmatrix} \cosh(2\mu z) & 0 & 0 & \sinh(2\mu z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(2\mu z) & 0 & 0 & \cosh(2\mu z) \end{pmatrix} \tag{49}$$

along the x direction. Here again the four-momentum of Equation (47) is invariant under rotations around the z axis.

The above four-by-four matrix corresponds to the two-by-two squeeze matrix of Equation (13) applicable to optical activities [5]. Thus, this squeeze matrix, together with the squeeze matrix of Equation (31), generate the little group for particles with space-like momentum.

Let us finally consider a massless particle with it four-momentum

$$(0, 0, p, p). \tag{50}$$

It is invariant under the rotation around the z axis. In addition, it is invariant under the transformation

$$\begin{pmatrix} 1 & 0 & -2\gamma & 2\gamma \\ 0 & 1 & 0 & 0 \\ 2\gamma & 0 & 1 - 2\gamma^2 & 2\gamma^2 \\ 2\gamma & 0 & -2\gamma^2 & 1 + 2\gamma^2 \end{pmatrix}. \tag{51}$$

This four-by-four matrix has a stormy history [8,10], but the bottom line is that it corresponds to the triangular matrix of Equation (37), and the variable γ performs gauge transformations.

It is interesting to note that optical activities can act as computational devices for the internal space–time symmetries of elementary particles.

5. Concluding remarks

Each human being has ten fingers. This is the origin of our decimal system. Vacuum tubes can do binary logic, and this is how the electronic computer was developed. Quantum two-level systems can do more than the vacuum tube can. This is why we are interested in quantum computers these days. Indeed, computers are based on the mathematical language spoken by nature.

Traditionally, polarization optics is known to produce the geometry of ellipse and that of the Poincaré sphere. It also produces the algebra of two-by-two and four-by-four matrices. In this paper, we started with rotations combined with asymmetric attenuations in optical activities. It was shown in this paper that the optical activity speaks the mathematical language of Wigner's little group dictating internal space–time symmetries of elementary particles.

We have used in this paper some mathematical methods not commonly seen in the conventional literature. In calculating the exponential form of a matrix, the usual procedure is to diagonalize the matrix by a unitary transformation. Then it is possible to write a Taylor expansion of the diagonal matrix.

What should we do if the matrix cannot be diagonalized by a unitary transformation? Let us go back to the G matrix of Equation (20). If $\gamma > \mu$, it was possible to bring the G the form

$$k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{52}$$

where the similarity transformation matrix of Equation (31) is not unitary. It is a symmetric squeeze matrix. In addition, we used the property

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{53}$$

to deal with the Taylor expansion. For $\mu > \gamma$, we used

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{54}$$

If $\mu = \gamma$, the G matrix becomes triangular, and

$$\begin{pmatrix} 0 & 0 \\ 2\gamma & 0 \end{pmatrix}^2 = 0. \tag{55}$$

The Taylor expansion truncates.

Using these properties of two-by-two matrices, we were able to deal with the problem even though not all of them can be diagonalized. The triangular matrix of Equation (55) is triangular and cannot be diagonalized. The matrix of Equation (54) can be diagonalized with the diagonal elements of 1 and −1. The two-by-two matrix of Equation (52) can also be diagonalized, but the eigenvalues are the imaginary numbers i and −i. However, the imaginary numbers are not too convenient for computer mathematics. Thus, we had to resort to the method presented in this paper.

References

[1] Azzam, R.M.A.; Bashara, N.M. *Ellisometry and Polarized Light*; Elsevier: Amsterdam, 1977.

[2] Brosseau, C. *Fundamentals of Polarized Light, A Statistical Optics Approach*; Wiley: New York, 1998.

[3] Theocaris, P.S.; Gdoutos, E.E. *Matrix Theory of Photoelasticity*; Springer-Verlag: Berlin, 1979.

[4] Han, D.; Kim, Y.S.; Noz, M.E. *J. Opt. Soc. Am. A* **1997,** *14,* 2290–2298.

[5] Han, D.; Kim, Y.S.; Noz, M.E. Noz, *Phys. Rev. E* **1997,** *56,* 6065–6076.

[6] Baskal, S.; Georgieva, E.; Kim, Y.S.; Noz, M.E. *J. Opt. B* **2004,** *6,* S455–S472.

[7] Wigner, E. *Ann. Math.* **1939,** *40,* 149–204.

[8] Kim, Y.S.; Noz, M.E. *Theory and Applications of the Poincaré Group*; Reidel: Dordrecht, 1986.

[9] Georgieva, E.; Kim, Y.S. *Phys. Rev. E* **2003,** *68,* 026606-1–6.

[10] Kim, Y.S.; Wigner, E.P. *J. Math. Phys.* **1990,** *31,* 55–60.