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Role of Quantum Optics in Synthesizing Quantum Mechanics and Relativity *

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Abstract

Two-photon states produce enough symmetry needed for Dirac's construction of the two-oscillator system which produces the Lie algebra for the O(3,2) space-time symmetry. This O(3,2) group can be contracted to the inhomogeneous Lorentz group which, according to Dirac, serves as the basic space-time symmetry for quantum mechanics in the Lorentz-covariant world. Since the harmonic oscillator serves as the language of Heisenberg's uncertainty relations, it is right to say that the symmetry of the Lorentz-covariant world, with Einstein's $E = mc^2$, is derivable from Heisenberg's uncertainty relations.

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1 Introduction

In 1963, Paul A. M. Dirac constructed the space-time symmetry of the deSitter group $O(3,2)$ [1]. This deSitter group can be contracted to the symmetry of the inhomogeneous Lorentz group which, according to Dirac, is the fundamental equation for quantum mechanics in the Lorentz-covariant world.

More recently, two-photon system became a prominent subject in physics. The purpose of this paper is to point out that Dirac's $O(3,2)$ system can be constructed from the two-photon systems of current interest. In 1976 [2], Yuen constructed the first formula for two-photon coherent states known as squeezed states. In 1986 [3], Yurke, McCall, and Klauder discussed two-photon interferometers exhibiting the U(1,1) and U(2) symmetries. If we combine these two-photon operators into one algebraic system, we end up with Dirac's $O(3,2)$ system.

Ever since Heisenberg declared his uncertainty relations in 1927, Paul A. M. Dirac was interested in whether quantum mechanics is consistent with Einstein's special relativity. In 1927 [4], Dirac notes that the c-number time-energy uncertainty relation causes a difficulty in making quantum mechanics Lorentz covariant. In 1945 [5], Dirac uses a Gaussian form with the time variable to construct a representation of the Lorentz group. However, he does not address the issue of the c-number nature of the time-energy uncertainty relation.

In 1949 paper in the special issue of the Reviews of Modern Physics in commemoration of Einstein's 70th Birthday [6], Dirac says that the task of constructing relativistic quantum mechanics is constructing a representation of the inhomogeneous Lorentz group. In the same paper, Dirac introduces the light-cone coordinate system telling the Lorentz boost is a squeeze transformation. In 1963, Dirac uses two coupled oscillators to construct the Lie algebra for the $O(3,2)$ deSitter group.

Indeed, Dirac made his lifelong efforts to synthesize quantum mechanics and special relativity. One hundred years ago, Bohr was interested in the electron orbit of the hydrogen atom. Einstein was in worrying about how things look to moving observers. Dirac was interested in this problem, but it was a metaphysical problem before 1960.

After 1950, the physics world started producing protons moving with speed comparable with that of light. In 1964, Gell-Mann produced his quark model telling the proton, like the hydrogen atom, is a quantum bound state of more fundamental particles called the quarks. In 1969, Feynman noted that the proton, when it moves with its speed close to that of light, appears as a collection partons with their peculiar properties.

Thus the Bohr-Einstein issue became the Gell-Mann-Feynman issue, as specified in Fig. 1. The oscillator representation of Dirac [5] allows us to use a circle in the longitudinal space-like and time-like coordinates. The light-cone coordinate system Dirac introduced in 1949 tells the Lorentz boost squeezed the oscillator circle into an ellipse as shown in Fig. 2. The question then is whether this effect of Lorentz squeeze can be observed in the real world.

In the papers written in and before 1949, Dirac was interested in combining two scientific disciplines into one. However, in 1963, by starting from the harmonic oscillators which represent Heisenberg's uncertainty relations, Dirac obtains the Lie algebra of the

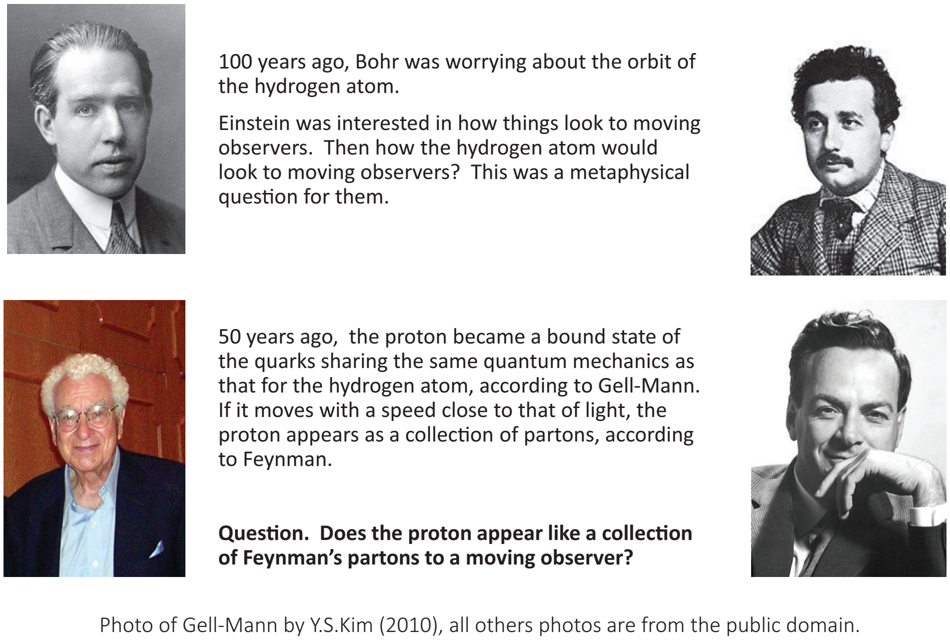

Figure 1: How the hydrogen atom look to moving observers? Fifty years later, this Bohr-Einstein issue becomes the Gell-Mann-Feynman issue. The issue is whether Feynman's partons are Gell-Mann's quarks viewed by a moving observer. This figure is from Ref. [10]

$O(3,2)$ deSitter group with ten generators, which is the Lorentz group applicable to three space-like and two time-like directions.

vspace1cm From his 1963 paper, we get a hint that this $O(3,2)$ group may be transformed into the inhomogeneous Lorentz group, which is the fundamental symmetry group for quantum mechanics in the Lorentz-covariant world according to Dirac [6]. This group also has ten generators. Six of them are for the Lorentz group and four of them are for space-time translations.

As in the case of Başkal [7], we show in this paper the inhomogeneous Lorentz group can be obtained from $O(3,2)$ via the procedure of group contractions introduced first by Inönü and Wigner in 1953 [8].

In Sec. 2, it is shown that two-photon states widely discussed in the current literature produce the ten generators for Dirac's two-oscillators system, which leads us to the Lie algebra of the $O(3,2)$ group. The five-by-five matrices for the ten generators of this group are also given. In Sec 3, the inhomogeneous Lorentz group is discussed. It is shown that this group can also be represented by five-by-five matrices. Five-by-five expressions are given also for ten generators of this group. In Sec. 4, the $O(3,2)$ croup is contracted to the inhomogeneous Lorentz group [9]. The four-momentum operators generated in this way corresponds to Einstein's $E = mc^2$.

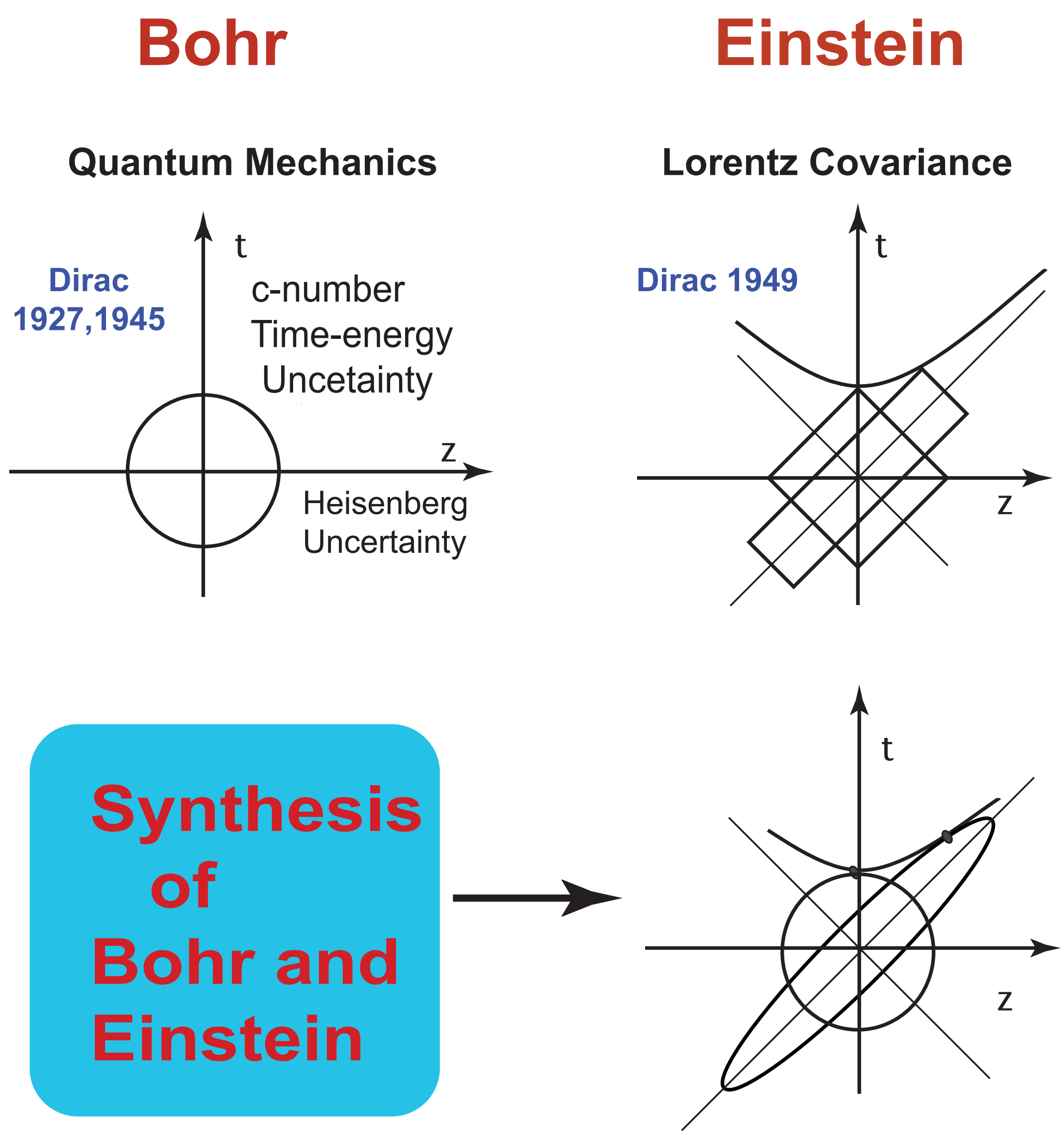

Figure 2: The hydrogen atom is a circle. The Lorentz boost is a squeeze transformation. If we combine them, the net effect is a squeezed circle.

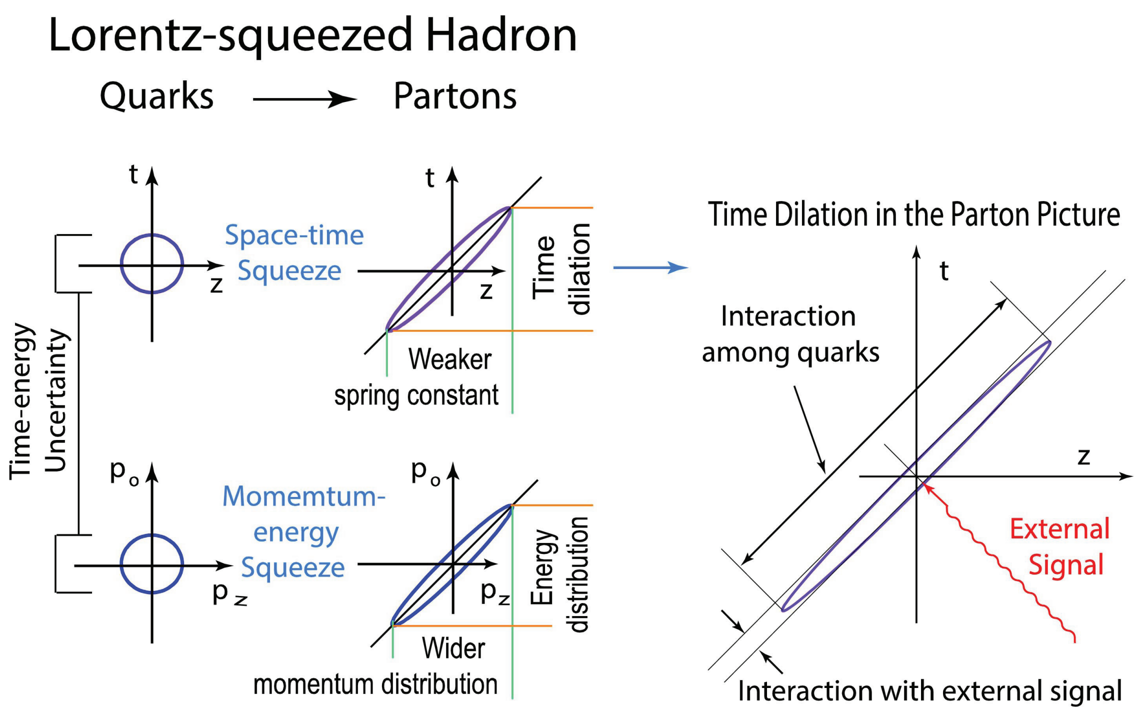

Figure 3: The crucial question is whether this squeezing effect can be observed in laboratories. This effects manifests itself through the wide-spread parton momentum distribution, short interaction time, partons as free light-like particles, as Feynman observed. This figure is from Ref. [10].

2 Dirac's Two-oscillator System from Quantum Optics

In 1963, Dirac published a paper entitled "A Remarkable Representation of the (3 + 2) deSitter Group" [1]. In this paper, he starts with two oscillators with the following step-up and step-down operators.

$$a_1 = \frac{1}{\sqrt{2}}\left(x_1 + iP_1\right), \qquad a_1^\dagger = \frac{1}{\sqrt{2}}\left(x_1 - iP_1\right),$$

$$a_2 = \frac{1}{\sqrt{2}}\left(x_2 + iP_2\right), \qquad a_2^\dagger = \frac{1}{\sqrt{2}}\left(x_2 - iP_2\right). \tag{1}$$

In terms of these operators, Heisenberg's uncertainty relations can be written as

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}. \tag{2}$$

with

$$x_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right), \qquad P_i = \frac{i}{\sqrt{2}}\left(a_i^\dagger - a_i\right), \tag{3}$$

With these sets of operators, Dirac constructed three generators of the form

$$J_1 = \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad J_2 = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right), \quad J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \tag{4}$$

and three more of the form

$$K_1 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$K_2 = +\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2\right),$$

$$K_3 = \frac{1}{2}\left(a_1^\dagger a_2^\dagger + a_1 a_2\right). \tag{5}$$

These J_i and K_i operators satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{6}$$

This set of commutation relations is identical to the Lie algebra of the Lorentz group where, J_i and L_i are three rotation and three boost generators respectively.

In addition, with the harmonic oscillators, Dirac constructed another set consisting of

$$Q_1 = -\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$Q_2 = -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2\right),$$

$$Q_3 = \frac{i}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right). \tag{7}$$

They then satisfy the commutation relations

$$[J_i, Q_j] = i\epsilon_{ijk} Q_k, \quad [Q_i, Q_j] = -i\epsilon_{ijk} J_k. \tag{8}$$

Together with the relation $[J_i, J_j] = i\epsilon_{ijk} J_k$ given in Eq.(6), J_i and Q_i and produce another set of closed commutation relations for the generators of the Lorentz group. Like K_i, the Q_i operators act as boost generators.

In order to construct a closed set of commutation relations for all the generators, Dirac introduced an additional operator

$$S_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right). \tag{9}$$

Then the commutation relations are

$$[K_i, Q_j] = -i\delta_{ij} S_0, \quad [J_i, S_0] = 0, \quad [K_i, S_0] = -iQ_i, \quad [Q_i, S_0] = iK_i. \tag{10}$$

Dirac then noted that these three sets of commutation relations given in Eqs. (6,8,10) constitute the Lie algebra for the group $O(3,2)$. This group is applicable to the five-dimensional space of (x, y, z, t, s), where x, y, z are for three space-like coordinates, and t and s are for time-like variables. The generators are therefore five-by-five matrices. These matrices are given in Table 1 and Table 2.

As Dirac stated in his paper [1], it is indeed remarkable that the two-oscillator system leads to the space-time symmetry of the $(3 + 2)$ deSitter group. Even more remarkable is that this two-oscillator system can be derived from quantum optics. In optics, a_i and $a_i^\dagger$ act as the annihilation and creation operators. For the two-photon system, i can be 1 or 2.

With these two sets of operators, it is possible to construct two-photon states. In 1976 [2], Yuen considered the two-photon state generated by

$$Q_3 = \frac{i}{2}\left(a_1^\dagger a_2^\dagger - a_1 a_2\right), \tag{11}$$

which leads to the two-mode coherent state known as the "squeezed state."

Later, in 1986 [3], Yurke *et al.* considered two-mode interferometers. In their study of two-mode states, they started with Q_3 given in Eq.(11). They then noted that, in one of their interferometers, the following two additional operators are needed.

$$K_3 = \frac{1}{2}\left(a_1^\dagger a_2^\dagger + a_1 a_2\right), \qquad S_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right). \tag{12}$$

The three Hermitian operators from Eq.(11) and Eq.(12) satisfy the commutation relations

$$[K_3, Q_3] = -iS_0, \qquad [Q_3, S_0] = iK_3, \qquad [S_0, K_3] = iQ_3. \tag{13}$$

Yurke *et al.* called this device the $SU(1,1)$ interferometer. The group $SU(1,1)$ is isomorphic to the $O(2,1)$ group or the Lorentz group applicable to two space-like and one time-like dimensions.

Table 1: Generators of the Lorentz group with three rotation and three boost generators applicable to the five-dimensional space of, where x, y, z are for space-like coordinates, t and s are for the time-like dimensions. These generators are totally separated from the s coordinate with zero elements on their fifth row and fifth column. The differential operators do not contain the s variable.

Generators	Differential	Matrix
J_1	$-i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
J_2	$-i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$	$\begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
J_3	$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$	$\begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
K_1	$-i\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)$	$\begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
K_2	$-i\left(y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
K_3	$-i\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 2: Four additional generators for the $O(3,2)$. Unlike those given in Table 2, the generators in this table have non-zero elements only in the fifth row and the fifth column. Every differential operator contains the s variable.

Generators	Differential	Matrix
Q_1	$-i\left(x\frac{\partial}{\partial s} + s\frac{\partial}{\partial x}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}$
Q_2	$-i\left(y\frac{\partial}{\partial s} + s\frac{\partial}{\partial y}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}$
Q_3	$-i\left(z\frac{\partial}{\partial s} + s\frac{\partial}{\partial z}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$
S_0	$-i\left(t\frac{\partial}{\partial s} - s\frac{\partial}{\partial t}\right)$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$

In addition, in the same paper [3], Yurke *et al.* discussed the possibility of constructing another interferometer exhibiting the symmetry generated by

$$J_1 = \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad J_2 = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right), \quad J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right). \tag{14}$$

These generators satisfy the closed set of commutation relations $[J_i, J_j] = i\epsilon_{ijk} J_k$, given in Eq.(6). This is the Lie algebra for the three-dimensional rotation group. Yurke *et al.* called this optical device the $SU(2)$ interferometer.

We are then led to ask whether it is possible to construct a closed set of commutation relations with the six Hermitian operators from Eq.(13) and Eq. (14). It is not possible. We have to add four additional operators, namely

$$\begin{aligned}
K_1 &= -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 - a_2^\dagger a_2^\dagger - a_2 a_2\right), \\
K_2 &= +\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 + a_2^\dagger a_2^\dagger - a_2 a_2\right), \\
Q_1 &= -\frac{i}{4}\left(a_1^\dagger a_1^\dagger - a_1 a_1 - a_2^\dagger a_2^\dagger + a_2 a_2\right), \\
Q_2 &= -\frac{1}{4}\left(a_1^\dagger a_1^\dagger + a_1 a_1 + a_2^\dagger a_2^\dagger + a_2 a_2\right).
\end{aligned} \tag{15}$$

There are now ten operators. They are precisely those ten Dirac constructed in his paper of 1963 [1].

It is indeed remarkable that Dirac's $O(3,2)$ algebra is produced by modern optics. This algebra produces the Lorentz group applicable to three space-like and two time-like dimensions.

3 Dirac's Forms of Relativistic Dynamics

In 1949 [6], Paul A. M. Dirac published a paper entitled "Forms of Relativistic Dynamics," where he stated that the construction of relativistic dynamics is to find a representation of the inhomogeneous Lorentz group [9]. This group is generated by three rotation generators, three boost generators, and four translation generators. If we use J_i and K_i for the rotation and boost generators respectively, and P_i and P_0 for the four momentum generators, they satisfy the following set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_k] = -i\epsilon_{ijk} J_k, \tag{16}$$

and

$$\begin{aligned}
&[P_i, J_k] = -i\epsilon_{ijk} J_k, \quad [P_i, K_k] = -i\epsilon_{ijk} K_k, \\
&[P_i, P_i] = 0, \quad [P_i, P_0] = 0, \quad [P_0, J_i] = [P_0, K_i] = 0.
\end{aligned} \tag{17}$$

There are ten generators, as in the case of the $O(3,2)$ group. Among them, the rotation and translation generators are Hermitian and correspond to observable dynamical variables, while the boost operators do not.

Table 3: Generators of translations in the four-dimensional Minkowski space. We are eventually interested in converting the four generators in the $O(3,2)$ group in Table 2 into the four translation generators.

Generators	Differential	Matrix
$Q_1 \rightarrow P_1$	$-i\frac{\partial}{\partial x}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$Q_2 \rightarrow P_2$	$-i\frac{\partial}{\partial y}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$Q_3 \rightarrow P_3$	$-i\frac{\partial}{\partial z}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$S_0 \rightarrow P_0$	$i\frac{\partial}{\partial t}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

As far as the Lorentz transformations are concerned, we can use four-by-four matrices. However, if we augment translations, we have to use the transformation of the type

$$\begin{pmatrix} x+x' \\ y+y' \\ z+z' \\ t+t' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & x' \\ 0 & 1 & 0 & 0 & y' \\ 0 & 0 & 1 & 0 & z' \\ 0 & 0 & 0 & 1 & t' \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix} . \tag{18}$$

This five-by-five matrix is constructed from translation generators from Table 3 according to

$$\exp\left\{-i\left(x'P_1 + y'P_2 + z'P_3 + t'P_0\right)\right\} . \tag{19}$$

In this five-by-five representation, the J_i and K_i generators can be written as the five-by-five matrices given in Table 1. The four translation generators are given in Table 3. There are ten generators, and the satisfy the Lie algebra of the inhomogeneous Lorentz given in Eq.(16) and Eq.(17). Table 3 indicates also that these translation operators can be obtained from Q_i and S_0 of the $O(3,2)$ group discussed in Sec. 2. We shall see how this happens in Sec. 4.

4 Contraction of O(3,2) to the Inhomogeneous Lorentz Group

We are interested in transforming the group $O(3,2)$ into the indigenous Lorentz group by contracting the s coordinate according to the group contraction procedure introduced first by E. Inönü and E. P. Wigner [8]. This procedure is applicable to contracting the three-dimensional rotation group into a two-dimensional Euclidean group. It is like the surface of the earth is flat for a limited area. The same Inönü-Wigner process can be used for contracting the Lorentz group into the group of three-dimensional rotations and three translations.

The idea of the present section is to use the same contraction procedure in order to convert $O(3,2)$ to the Lorentz group applicable to the four-dimensional Minkowski space and four translations.

For this purpose, we cab make the s coordinate continuously smaller to zero in the limit:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ s \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x \\ y \\ x \\ t \\ 0 \end{pmatrix} , \tag{20}$$

The contracted vector with $s = 0$ remains invariant under the inverse transformation.

Let us use the notation $C(\epsilon)$ for the five-by-five matrix given in Eq.(20). This matrix commutes with J_i and K_i given in Table 1. As for those in Table 2, the same transformation on the matrix Q_1 is

$$C\ Q_1\ C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & i/\epsilon \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i\ \epsilon & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

which in, in the limit of small ϵ, becomes

$$C\ Q_1\ C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & i/\epsilon \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

This matrix had only one non-zero element. Thus inverse of this transformation leads to

$$\epsilon\ C\ Q_1\ C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

This is precisely the matrix for the translation operator given in Table 3. We can carry out similar procedures for other items in the same table to complete the contraction of Dirac's $O(3,2)$ into the inhomogeneous Lorentz group.

Special relativity and quantum mechanics have been and still are two major physical theories formulated during the past century. It is gratifying to note that these theories are derivable from the same mathematical base, namely the mathematics of two coupled oscillators.

5 Acknowledgments

This report is, in part, based on the paper I published with Sibel Başkal and Marilyn Noz [7]. I would like thank them for their many years of collaboration.

Dirac's 1963 paper on the $O(3,2)$ group is not widely known, and I am the only one talking about this paper constantly. There is a good reason. I met him in October of 1962 after he completed this article.

Dirac did not talk to too many people. How did I meet him? I finished my PhD degree at Princeton in 1961 and stayed there for one more year as a post-doc before becoming an assistant professor at the Univ. of Maryland in 1962. At that time, the chairman of the physics department was John S. Toll, and he was an ambitious man. He invited Dirac in October of 1962 for one week. Since I was the youngest faculty member in his department, Toll assigned me as Dirac's personal assistant.

At that time, I was publishing my papers acceptable to the American physics community. I had to write my papers starting from the premise that the physics starts from singularities in the two-dimensional complex plane, but I knew that I was writing useless (if not wrong) papers. Dirac indeed taught me how to do physics: synthesize quantum mechanics and relativity. I was like Nicodemus meeting Jesus (story from the Gospel of John in the New Testament). I was born again.

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