Lorentz-Invariant Minimum Uncertainty Product

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Due to the circumstances beyond his control, the author of this paper was not able to proof-read the galley, in spite of the journal's policy to obtain a proof-read copy prior to publication.

Updated References

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It is shown that the covariant harmonic oscillator wave functions produce a Lorentz-invariant minimum uncertainty product in the light-cone coordinate system. This result enables us to interpret the Lorentz invariance of Planck's constant.

I. INTRODUCTION

Planck's constant is the most important physical quantity in quantum mechanics. The original concept of this constant was developed by Planck in connection with his effort to understand black-body radiation. The fact that this quantity serves as a proportionality constant between the photon energy and frequency was decisively demonstrated by Einstein's photoelectric effect.

In nonrelativistic quantum mechanics, Planck's constant represents the ultimate numerical accuracy in measurement processes. In the Schrödinger picture, this constant appears as the uncertainty exhibited by a Gaussian or the ground-state harmonic oscillator wave function. All other wave functions have greater uncertainties.

The purpose of this note is to examine whether the numerical value of Planck's constant defined in the Schrödinger picture remains the same for observers in different Lorentz frames of reference. This is a serious problem because the wave function in one Lorentz frame appears deformed to the observer in another frame. The concept of Planck's constant as the ultimate numerical accuracy in measurement will be a Lorentz invariant only if the observer in a moving Lorentz frame can extract the same numerical value for the minimum uncertainty from the Gaussian wave function in the rest frame.

In Sec. II, we examine the covariant probability interpretation which can be given to the relativistic harmonic oscillator formalism developed by Kim and Noz. [1,2] We also present a more precise interpretation than is available in the existing literature.

In Sec. III, a set of Lorentz-invariant minimum uncertainty products is presented. This result of course enables us to interpret the Lorentz invariance of Planck's constant.

In Sec. IV, we discuss the significance of the result obtained.

II. COVARIANT PROBABILITY INTERPRETATION

As was stated in Sec. I, the concept of minimum uncertainty depends crucially on the probability interpretation which can be given to the Gaussian wave function. Since the wave function with a covariant probability interpretation is so new, [3] we shall briefly review here the published formalism. We shall then present a probability interpretation which is more precise than is available in previous publications. [1,2]

Let us consider a system of two quarks bound together by a harmonic oscillator force. For this system, Feynman *et. al.* [4] proposed the relativistic wave equation

$$\{2[\Box_1+\Box_2]-\frac{\omega^2}{16}(x_1-x_2)^2+m^2_0\}\phi(x_1,x_2)=0 \tag{1}$$

as a possible alternative to the conventional Feynman diagram approach to strong interaction dynamics. ω in the above expression is the spring constant. If we make the variable transformations

$$X=\frac{1}{2}(x_1+x_2),\quad x=\frac{1}{2\sqrt{2}}(x_1-x_2), \tag{2}$$

then Eq. (1) is separable, and $\phi(x_1,x_2)$ can be written as

$$\phi(x_1,x_2)=f(X)\Psi(x), \tag{3}$$

where $f(X)$ and $\Psi(x)$ satisfy the following equations respectively.

$$(\frac{\partial^2}{\partial X^2_\mu}+m^2_0+\lambda)f(X)=0, \tag{4}$$

$$\frac{1}{2}\left[\frac{\partial^2}{\partial x^2_\mu}-\omega^2x^2_\mu\right]\Psi(x)=\lambda\Psi(x). \tag{5}$$

The differential equation of Eq. (4) in a Klein-Gordon equation in the X variable which can be regarded as the hadronic coordinate. The physics of the Klein-Gordon equation is well known. [5] Eq. (5) is a relativistic harmonic oscillator equation for the quarks, and x_μ denotes the space and time separations of these bound quarks.

Eq. (5) is a hyperbolic partial differential equation. Its solutions take many different forms depending on boundary and subsidiary conditions. In their recent paper, Kim and Noz [1] constructed a set of solutions which are localized in a bounded space-time region. They observed that Eq. (5) is separable in terms of the $x, y, z,$ and t variables, and that this equation is also separable in terms of the following Lorentz-transformed variables.

$$\begin{aligned} &x'=x,\quad y'=y,\\ &z'=(1-\beta^2)^{-1/2}(z-\beta t),\\ &t'=(1-\beta^2)^{-1/2}(t-\beta z). \end{aligned} \tag{6}$$

where β is the velocity parameter of the hadron along the z-axis. The normalizable solution then becomes

$$\Psi_\beta(x)=N_{lmn}H_l(x')H_m(y')H_n(z') \times\exp\left\{-\frac{\omega}{2}(x'^2+y'^2+z'^2+t'^2)\right\} \tag{7}$$

where $H_l(x')$ is the Hermite polynomial, and N_{lmn} is the normalization constant. The subsidiary condition

$$P^\mu(\omega x_\mu+\frac{\partial}{\partial x^\mu})\Psi_\beta(x)=0 \tag{8}$$

eliminates time-like excitations in the t' variable. P^μ is the four-momentum of the hadron having the velocity parameter β. The wave function of Eq. (7) can be written in a covariant form. [1]

We now present probability interpretation which is more precise than is available in previous publications. [1,2] Since the t' variable is separable and since there are no time-like oscillations, the wave function of Eq. (7) involves only non-relativistic quantum mechanics if all the wave functions have the same β parameter. The harmonic oscillator wave functions in this case are orthonormal and complete. For two different values of β, it was shown by Ruiz [6] that the orthogonality relations still hold and that the Lorentz-invariant inner product contracts in proportion to $(\sqrt{1-\alpha^2})^{n+1}$, where α is the relativistic velocity difference and n is the excitation along the z-axis. According to this result, the ground-state wave function with one half-wave and no node contracts like a rigid rod. The n-th excited-state wave function can be regarded as the ground-state wave function multiplied by n step-up operators. Each step-up operator transforms the same as the z-coordinate and thus like a rigid rod. Therefore the Lorentz contraction of the n-th excited state should be $(\sqrt{1-\alpha^2})^{n+1}$. This is exactly what the covariant harmonic oscillators produce. These orthogonality and Lorentz contraction properties enable us to attach a covariant probability interpretation to the harmonic oscillator wave functions.

The crucial difference between the covariant oscillator and non-relativistic quantum mechanics is the existence of the t' variable in the covariant formalism. The formalism allows a ground-state uncertainty but does not allow excitation along the t' axis. The most important question then is whether there is experimental evidence that proves the existence of this peculiar uncertainty.

Although the time-energy uncertainty does not exist in any of the formalisms of non-relativistic quantum mechanics, it is known to exist in nature through the relation between the width and life-time of resonances. [7] Recently, the harmonic oscillator formalism with its built-in time-energy uncertainty has been used to explain the Lorentz contraction phenomena in hadronic physics. [1, 2, 8] Perhaps the most dramatic experimental indication is in Feynman's parton picture. [9] Feynman observed that a fast-moving hadron can be regarded as a collection of partons whose peculiar properties cannot be explained by local field theory. However, the covariant harmonic oscillator which we discussed above can answer all of the irritating questions in the parton picture [10]

III. MINIMUM UNCERTAINTY PRODUCTS

With the above preparation, we can now attack the main problem of this paper. We are here discussing a ground-state wave function attached to a Lorentz frame which moves along the z direction with velocity parameter β. The question then is how this moving wave function exhibits its minimum uncertainty to the observer in the rest frame.

If $\beta=0$, then the question is trivial. If $\beta\neq0$, the ground-state wave function can be written as [11]

$$\Psi_\beta(x)=\left(\frac{\omega}{\pi}\right)\exp\left\{-\frac{\omega}{2}\left[x^2+y^2+\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{z+t}{\sqrt{2}}\right)^2+\left(\frac{1+\beta}{1-\beta}\right)\left(\frac{z-t}{\sqrt{2}}\right)^2\right]\right\} \quad (9)$$

The quadratic form in the exponent is diagonal in the light-cone variables. [12] The width of the wave function depends on the parameter β. Let us next consider the momentum wave function by making a Fourier transformation

$$\phi_\beta(p)=\left(\frac{1}{2\pi}\right)\int d^4x\Psi_\beta(x)\exp(-ip\cdot x) \quad (10)$$

The momentum wave function then takes the form

$$\phi_\beta(p)=\left(\frac{1}{\pi\omega}\right)\exp\left\{-\frac{1}{2\omega}\left[p_x^2+p_y^2+\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{p_z+p_0}{\sqrt{2}}\right)^2+\left(\frac{1+\beta}{1-\beta}\right)\left(\frac{p_z-p_0}{\sqrt{2}}\right)^2\right]\right\} \quad (11)$$

The momentum wave function is also diagonal in the light-cone variables.

If we define

$$\xi=\frac{z+t}{\sqrt{2}},\quad \eta=\frac{z-t}{\sqrt{2}}, \quad (12)$$

then

$$\frac{\partial}{\partial\xi}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right),$$
$$\frac{\partial}{\partial\eta}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right) \quad (13)$$

If we define

$$p_z = -i\frac{\partial}{\partial z}, \quad p_o = i\frac{\partial}{\partial t}, \tag{14}$$

then

$$p_\xi = i\frac{\partial}{\partial \xi} = \frac{1}{\sqrt{2}}(p_z - p_o),$$
$$p_\eta = i\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}(p_z + p_o) \tag{15}$$

In terms of these light-cone variables, we can now write the following Lorentz-invariant uncertainty products.

$$\langle \xi^2 \rangle \langle p_\xi^2 \rangle = \frac{\hbar^2}{4},$$
$$\langle \eta^2 \rangle \langle p_\eta^2 \rangle = \frac{\hbar^2}{4} \tag{16}$$

These β-independent relations enable us to interpret the Lorentz invariance of Planck's constant. In the rest frame where $\beta=0$, the oscillator wave function gives the minimum uncertainty products for space and time variables separately. However, before we boost the system, we have to transform the space-time variables into the light-cone coordinate system in order to maintain the minimum in the uncertainty products.

IV. CONCLUDING REMARKS

In this paper, we discussed first a precise probability interpretation which can be given to the relativistic harmonic oscillator wave functions. Using the ground-state wave function, we then established the Lorentz-invariant concept of Planck's constant as the ultimate limit of accuracy for the quantum measurement processes.

In collaboration with M. E. Noz, this author undertook the task of constructing a set of relativistic bound-state wave functions which can carry a covariant probability interpretation. This task was not accomplished in a single paper. The first attempt was reported in Ref. 1. Ref. 2 contains a major improvement over the previous paper. The present paper completes the task of constructing the first set of covariant bound state wave functions since the development of quantum mechanics.

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