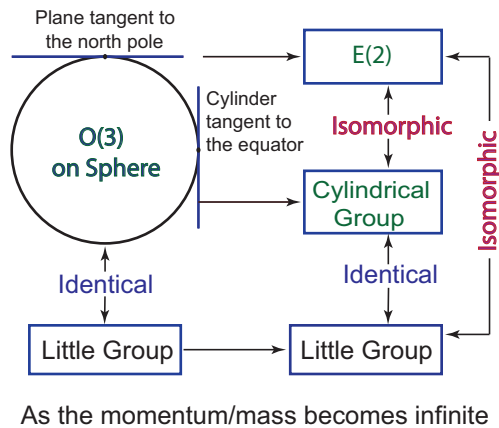# Wigner's Little Groups

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Abstract

Wigner's little groups are subgroups of the Lorentz group dictating the internal space-time symmetries of massive and massless particles. The little group for the massive particle is like O(3) or the three-dimensional rotation group, and the little group for the massless particle is E(2) or the two-dimensional Euclidean group consisting of rotations and translations on a two-dimensional plane. While the geometry of the O(3) symmetry is familiar to us, the geometry of the flat plane cannot explain the E(2)-like symmetry for massless particles. However, the geometry of a circular cylinder can explain the symmetry with the helicity and gauge degrees of freedom. It is shown that the cylindrical group is like E(2) and thus like the little group for the massless particle. While Wigner discussed the O(3)-like little group for the massive particle at rest, it is possible to Lorentz-boost this rotation matrix. It is shown further that the E(2)-like symmetry of the massless particle can be obtained as a zero-mass limit of O(3)-like symmetry for massive particles. It is shown further that the polarization of massless neutrinos is a consequence of gauge invariance, while the symmetry of massive neutrinos is still like O(3).

As the momentum/mass becomes infinite

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1   Introduction

In his 1939 paper [1], Wigner considered the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. These subgroups are called Wigner's little groups and dictate the internal space-time symmetries in the Lorentz-covariant world. He observed first that a massive particle at rest has three rotational degree of freedom leading to the concept of spin. Thus the little group for the massive particle is like O(3).

Wigner observed also that a massless particle cannot be brought to its rest frame, but he showed that the little group for the massless particle also has three degrees of freedom, and that this little is locally isomorphic to the group E(2) or the two-dimensional Euclidean group. This means that generators of this little group share the same set of closed commutation relations with that for two-dimensional Euclidean group with one rotational and two translational degrees of freedom.

It is not difficult to associate the rotational degree of freedom of E(2) to the helicity of the massless particle. However, what is the physics of the those two translational degrees of freedom? Wigner did not provide the answer to this question in his 1939 paper [1]. Indeed, this question has a stormy history and the issue was not completely settled until 1990 [2], fifty one years after 1939.

In this report, it is noted first that the Lorentz group has six generators. Among them, three of them generate the rotation subgroup. It is also possible to construct three generators which constitute a closed set of commutations relations identical to that for the E(2) group. However, it is also possible to construct the cylindrical group with one rotational degree of freedom and two degrees freedom both leading to up-down translational degrees freedom. These two translational degrees freedom correspond to one gauge degree of freedom for the massless particle [3].

While the O(3)-like and E(2)-like little groups are different, it is possible to derive the latter as a Lorentz-boosted O(3)-like little group in the infinite-momentum limit. It is shown then that the two rotational degrees of freedom perpendicular momentum become one gauge degree of freedom [4].

It is noted that the E(2)-like symmetry for the massless spin-1 particle leads to its helicity and gauge degree of freedom. Likewise, there is a gauge degree of freedom for the massless spin-1/2 particle. However, the requirement of gauge invariance leads to the polarization of massless neutrinos [5, 6, 7].

In Sec. 2, we introduce Wigner's little groups for massive and massless particles. In Sec. 3, the same logic is developed for spin-half particles. It is shown that the polarization of massless neutrinos is a consequence of gauge invariance. In Sec. 4, it noted that the four-vectors can be constructed from the spinors in the Lorentz-covariant world. It is shown that the gauge transformation for the spin-1/2 particle manifest itself as the gauge transformation in the world of the four-vectors, which is more familiar to us. In Sec. 5,

it is shown that the E(2)-like little group for massless particles can be obtained as the infinite-momentum limit of the O(3)-like little group.

In Sec. 6, we noted how the Wigner's little groups dictate the internal space-time symmetries of massive and massless particles. Thus, the same formalism can be used for moving hydrogen atoms, or bound states in the Lorentz-covariant world. We give a brief review of the progress made along this direction.

2 Wigner's little groups

If we use the four-vector convention $x^\mu = (x, y, z, t)$, the generators of rotations around and boosts along the z axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{1}$$

respectively. We can also write the four-by-four matrices for J_1 and J_2 for the rotations around the x and y directions, as well as K_1 and K_2 for Lorentz boosts along the x and y directions respectively [6]. These six generators satisfy the following set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{2}$$

This closed set of commutation relations is called the Lie algebra of the Lorentz group. The three J_i operators constitute a closed subset of this Lie algebra. Thus, the rotation group is a subgroup of the Lorentz group.

In addition, Wigner in 1939 [1] considered a subgroup generated by

$$J_3, \qquad N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{3}$$

These generators satisfy the closed set of commutation relations

$$[N_1, N_2] = 0, \qquad [J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_1. \tag{4}$$

As Wigner observed in 1939 [1], this set of commutation relations is just like that for the generators of the two-dimensional Euclidean group with one rotation and two translation generators, as illustrated in Fig. 1. However, the question is what aspect of the massless particle can be explained in terms of this two-dimensional geometry.

Indeed, this question has a stormy history, and was not answered until 1987. In their paper of 1987 [3], Kim and Wigner considered the surface of a circular cylinder as shown in Fig. 1. For this cylinder, rotations are possible around the z axis. It is also possible
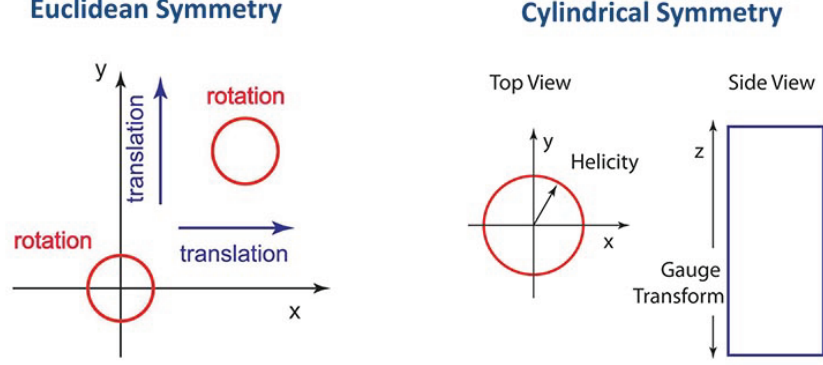
Figure 1: Transformations of the $E(2)$ group and the cylindrical group. They share the same Lie algebra, but only the cylindrical group leads to a geometrical interpretation of the gauge transformation.

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to make translations along the z axis as shown in Fig. 1. We can write these generators as

$$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \tag{5}$$

applicable to the three-dimensional space of (x, y, z). They then satisfy the closed set of commutation relations

$$[Q_1, Q_2] = 0, \qquad [L_3, Q_1] = iQ_2, \qquad [L_3, Q_2] = -iQ_1. \tag{6}$$

which becomes that of Eq.(4) when Q_1, Q_2, and L_3 are replaced by N_1, N_2, and J_3 of Eq.(3) respectively. Indeed, this cylindrical group is locally isomorphic to Wigner's little group for massless particles.

Let us go back to the generators of Eq.(3). The role of J_3 is well known. It is generates rotations around the momentum and corresponds to the helicity of the massless particle. The N_1 and N_2 matrices take the form [6]

$$N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{7}$$

The transformation matrix is

$$D(u, v) = \exp\left\{-i\left(uN_1 + vN_2\right)\right\} = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \tag{8}$$

4

In his 1939 paper [1], Wigner observed that this matrix leaves the four-momentum of the massless particle invariant as can be seen from

$$\begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1-(u^2+v^2)/2 & (u^2+v^2)/2 \\ u & v & -(u^2+v^2)/2 & 1+(u^2+v^2)/2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_3 \end{pmatrix}, \tag{9}$$

but he never attempted to apply this matrix to the photon four-potential.

It is interesting to note that Kuperzstych in 1976 noted that this form in applicable to the four-potential while making rotation and boosts whose combined effects do not change the four four-momentum [8] of the photon. In 1981, Han and Kim carried out the same calculation within the framework of Wigner's little group [9]. Kuperzstych's conclusion was that the four-by-four matrix of Eq.(8) performs a gauge transformation when applied to the photon four-potential, and Han and Kim arrived at the same conclusion. Let us see how this happens.

Ler us next consider the electromagnetic wave propagating along the z direction:

$$A^\mu(z,t) = (A_1, A_2, A_3, A_0)e^{i\omega(z-t)}, \tag{10}$$

and appy the $D(U,v)$ matrix to this electromagneti four-vector:

$$\begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1-(u^2+v^2)/2 & (u^2+v^2)/2 \\ u & v & -(u^2+v^2)/2 & 1+(u^2+v^2)/2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix}, \tag{11}$$

which becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} - (A_3 - A_0)\begin{pmatrix} u \\ v \\ (u^2+v^2)/2 \\ (u^2+v^2)/2 \end{pmatrix}. \tag{12}$$

If the four-vector satisfies the Lorentz condition $A_3 = A_0$, this expression becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ uA_1 + vA_3 \\ uA_1 + vA_3 \end{pmatrix}. \tag{13}$$

The net effect is an addition of the same quantity to the longitudinal and time-like components while leaving the transverse components invariant. Indeed, this is a gauge transformation.
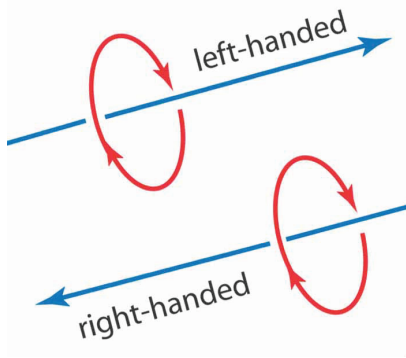
Figure 2: Polarization of massless neutrinos. Massless neutrinos are left-handed, while anti-neutrinos are right-handed. This is a consequence of gauge invariance.

3   Spin-1/2 particles

Let us go back to the Lie algebra of the Lorentz group given in Eq.(2). It was noted that there are six four-by-four matrices satisfying nine commutation relations. It is possible to construct the same Lie algebra with six two-by-two matrices [6]. They are

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad K_i = \frac{i}{2}\sigma_i, \tag{14}$$

where σ_i are the Pauli spin matrices. While J_i are Hermitian, K_i are not. They are anti-Hermitian. Since the Lie algebra of Eq.(2) is Hermitian invariant, we can construct the same Lie algebra with

$$J_i = \frac{1}{2}\sigma_i, \quad \text{and} \quad \dot{K}_i = -\frac{i}{2}\sigma_i. \tag{15}$$

This is the reason why the four-by-four Dirac matrices can explain both the spin-1/2 particle and the anti-particle.

Thus the most general form of the transformation matrix takes the form

$$T = \exp\left(-\frac{i}{2}\sum_i \theta_i \sigma_i + \frac{1}{2}\sum_i \eta_i \sigma_i\right), \tag{16}$$

and this transformation matrix is applicable to the spinors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{17}$$

In addition, we have to consider the transformation matrices

$$\dot{T} = \exp\left(-\frac{i}{2}\sum_i \theta_i \sigma_i - \frac{1}{2}\sum_i \eta_i \sigma_i\right), \tag{18}$$

applicable to

$$\dot\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \dot\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{19}$$

With this understanding, let us go back to the Lie algebra of Eq.(2). Here again the rotation generators satisfy the closed set of commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad \left[\dot J_i, \dot J_j\right] = i\epsilon_{ijk}\dot J_k. \tag{20}$$

These operators generate the rotation-like $SU(2)$ group, whose physical interpretation is well known, namely the electron and positron spins.

Here also we can consider the $E(2)$-like subgroup generated by

$$J_3, \qquad N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{21}$$

The N_1 and N_2 matrices take the form

$$N_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

On the other hand, in the "dotted" representation,

$$\dot N_1 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \qquad \dot N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

There are therefore two different D matrices:

$$D(u,v) = exp\{-(iuN_1 + ivN_2)\} = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \tag{24}$$

and

$$\dot D(u,v) = exp\left\{-\left(iu\dot N_1 + iv\dot N_2\right)\right\} = \begin{pmatrix} 1 & 0 \\ u + iv & 1 \end{pmatrix}. \tag{25}$$

These are the gauge transformation matrices applicable to massless spin-1/2 particles [5, 6].

Here are talking about the Dirac equation for with four-component spinors.

The spinors χ_+ and $\dot\chi_-$ are gauge-invariant since

$$D(u,v)\chi_+ = \chi_+, \quad \text{and} \quad \dot D(u,v)\dot\chi_- = \dot\chi_-. \tag{26}$$

As for χ_- and $\dot\chi+$,

$$D(u,v)\chi_- = \chi_- + (u - iv)\chi_+,$$

$$\dot D(u,v)\dot\chi_+ = \dot\chi_+ + (u + iv)\dot\chi_-. \tag{27}$$

They are not invariant under the D transformations, and they are not gauge-invariant. Thus, we can conclude that the polarization of massless neutrinos is a consequence of gauge invariance, as illustrated in Fig. 2.

4 Four-vectors from the spinors

We are familiar with the way in which the spin-1 vector is constructed from the spinors in non-relativistic world. We are now interested in constructing four-vectors from these spinors. First of all, with four of the spinors given above, we can start with the products.

$$\chi_i\chi_j, \quad \chi_i\dot{\chi}_j, \quad \dot{\chi}_i\chi_j, \quad \dot{\chi}_i\dot{\chi}_i. \tag{28}$$

resulting in spin-0 scalars and four-vectors and four-by-four tensors for the spin-1 states [6]. The four-vector can be constructed from the combinations $\chi_i\dot{\chi}_j$ and $\dot{\chi}_i\chi_j$.

Among them, let us consider the combinations, let us consider the four resulting from $\dot{\chi}_i\chi_j$. Among them, As far as the rotation subgroup is concerned, $\dot{\chi}_+\chi_+$, and $\dot{\chi}_-\chi_-$ are like $-(x+iy)$ and $(x-iy)$ respectively, and and invariant under Lorentz boosts along the z direction. In addition, we should consider

$$\frac{1}{2}\left(\dot{\chi}_-\chi_+ + \dot{\chi}_+\chi_-\right), \quad \text{and} \quad \frac{1}{2}\left(\dot{\chi}_-\chi_+ - \dot{\chi}_+\chi_-\right), \tag{29}$$

which are invariant under rotations around the z axis. When the system boosted along the z direction, these combinations are transformed like z and t directions respectively.

With these aspects in mind, let us consider the matrix

$$M = \begin{pmatrix} \dot{\chi}_-\chi_+ & \dot{\chi}_-\chi_- \\ -\dot{\chi}_+\chi_+ & -\dot{\chi}_+\chi_- \end{pmatrix}, \tag{30}$$

and write the transformation matrix T of Eq.(16)as

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{with} \quad \det(T) = 1. \tag{31}$$

If four matrix elements are complex numbers, there are eight independent parameters. However, the condition $\det(T) = 1$ reduces this number to six. The Lorentz group starts with six degrees of freedom.

It is then possible to write the four-vector (x, y, z, t) as

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \tag{32}$$

with its Lorentz-transformation property

$$X' = T \ X \ T^\dagger, \tag{33}$$

The four-momentum can also be written as

$$P = \begin{pmatrix} p_0+p_3 & p_1-ip_2 \\ p_1+ip_2 & p_0-p_3 \end{pmatrix}, \tag{34}$$

with the the transformation property same as that for X given in Eq.(33).

With this understanding, we can write the photon four-potential as

$$A = \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix} \tag{35}$$

Let us go back the two-by-two matrices $D(u,v)$ and $\dot{D}(u,v)$ given in Eqs.(24,25). We said there that they perform gauge transformations on massless neutrinos. It is indeed gratifying to note that they also lead to the gauge transformation applicable to the photon four-potential.

$$D(u,v)AD^\dagger(u,v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u + iv & 1 \end{pmatrix}. \tag{36}$$

This results in

$$\begin{pmatrix} A_0 + A_3 + 2(uA_1 + vA_2) & A_1 - iA_2 \\ A_1 + iA_2+ & A_0 - A_3 \end{pmatrix} + (A_0 - A_3) \begin{pmatrix} u^2 + v^2 & u - iv \\ u + iv & 1 \end{pmatrix}. \tag{37}$$

If we apply the Lorentz condition $A_0 = A_3$, this matrix becomes

$$\begin{pmatrix} 2A_z + 2(uA_1 + vA_2) & A_1 - iA_2 \\ A_1 + iA_2 & 0 \end{pmatrix}. \tag{38}$$

This result is the same as the gauge transformation in the four-by-four representation given in Eq.(13).

5 Massless particle as a limiting case of massive particle

In this two-by-two representation, the Lorentz boost along the positive direction is

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \tag{39}$$

the rotation around the y axis is

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{40}$$

Then, the boosted rotation matrix is

$$B(\eta)R(\theta)B(-\eta) = \begin{pmatrix} \cos(\theta/2) & -e^\eta \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{41}$$

If η becomes very large, and this matrix is to remain finite, θ has to become very small, and this expression becomes [7]

$$\begin{pmatrix} 1 - r^2 e^{-2\eta}/2 & r \\ -re^{-2\eta} & 1 - r^2 e^{-2\eta}/2 \end{pmatrix}. \tag{42}$$

with

$$r = -\frac{1}{2}\theta e^{\eta}. \tag{43}$$

This expression becomes

$$D(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}. \tag{44}$$

In this two-by-two representation, the rotation around the z axis is

$$Z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \tag{45}$$

respectively. Thus

$$D(u, v) = Z(\phi)D(r)Z^{-1}(\phi), \tag{46}$$

which becomes

$$D(u, v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \tag{47}$$

with

$$u = r\cos\phi, \quad \text{and} \quad v = r\sin\phi, \tag{48}$$

Here, we have studied how the little group for the O(3)-like little group the massive particle becomes the E(2)-like little group for the massless particle in the infinite-η limit. What does this limit mean physically? The parameter η can be derived from the speed of of the particle. We know $tanh(\eta) = v/c$, where v is the speed of the particle. Then

$$\tanh\eta = \frac{p}{\sqrt{m^2 + p^2}}, \tag{49}$$

where m and p are the mass and the momentum of the particle respectively. If m is much smaller than $/p$,

$$e^{\eta} = \frac{\sqrt{2}p}{m}, \tag{50}$$

which becomes large when m becomes very small. Thus, the limit of large η means the zero-mass limit.

Let us carry our the same limiting process for the four-by-four representation. From the generators of the Lorentz group, it is possible to construct the four-by-four matrices for rotations around the y axis and Lorentz boosts along the z axis as [6]

$$R(\theta) = \exp\left(-i\theta J_2\right), \quad \text{and} \quad B(\eta) = \exp\left(-i\eta K_3\right), \tag{51}$$

10

Table 1: One little group for both massive and massless particles. Einstein's special relativity gives one relation for both. Wigner's little group unifies the internal space-time symmetries for massive and massless particles which are locally isomorphic to O(3) and E(2) respectively. This table suggests a Lorentz-covariant picture of the bound state as manifested in the quark and parton models. This table is from Ref. [10].

	Massive, Slow	COVARIANCE	Massless, Fast
Energy-Momentum	$E = p^2/2m$	Einstein's $E = \sqrt{p^2 + m^2}$	$E = p$
Internal space-time symmetry	S_3 S_1, S_2	Wigner's Little Group	S_3 Gauge Transformations
Moving H-atom	Gell-Mann's Quark Model	Covariant Bound State	Feynaman's Parton Model

respectively. The Lorentz-boosted rotation matrix is $B(\eta)R(\theta)B(-\eta)$ which can be written as

$$
\begin{pmatrix}
\cos\theta & 0 & (\sin\theta)\cosh\eta & -(\sin\theta)\sinh\eta \\
0 & 1 & 0 & 0 \\
-(\sin\theta)\cosh\eta & 0 & \cos\theta - (1-\cos\theta)\sinh^2\eta & (1-\cos\theta)(\cosh\eta)\sinh\eta \\
-(\sin\theta)\cosh\eta & 0 & -(1-\cos\theta)(\cosh\eta)\sinh\eta & \cos\theta + (1-\cos\theta)\cosh^2\eta
\end{pmatrix}. \tag{52}
$$

While $\tanh\eta = v/c$, this boosted rotation matrix becomes a transformation matrix for a massless particle when η becomes infinite. On the other hand, it the matrix is to be finite in this limit, the angle θ has to become small. If we let $r = -\frac{1}{2}\theta e^\eta$ as given in Eq.(43), this four-by-four matrix becomes

$$
\begin{pmatrix}
1 & 0 & -r & r \\
0 & 1 & 0 & 0 \\
r & 0 & 1 - r^2/2 & r^2/2 \\
r & 0 & -r^2/2 & 1 + r^2/2
\end{pmatrix}. \tag{53}
$$

This is the Lorentz-boosted rotation matrix around the y axis. However, we can rotate this y axis around the z axis by ϕ. Then the matrix becomes

$$
\begin{pmatrix}
1 & 0 & -r\cos\phi & r\cos\phi \\
0 & 1 & -r\sin\phi & r\sin\phi \\
r\cos\phi & r\sin\phi & 1 - r^2/2 & r^2/2 \\
r\cos\alpha & r\sin\phi & -r^2/2 & 1 + r^2/2
\end{pmatrix}. \tag{54}
$$

This matrix becomes $D(u, v)$ of Eq.(8), if replace $r\cos\phi$ and $r\sin\phi$ with u and v respectively, as given in Eq.(48).

6 Historical Implications

For many years, the major complaint against Wigner's 1939 paper had been that his little groups could not explain the Maxwell field. The electromagnetic field propagating with it electric and magnetic field perpendicular to each other and perpendicular to the momentum. This issue has been settled. Thus, Wigner's little group is now the symmetry group dictating the internal space-time symmetries.

In his 1939 paper [1], Wigner discussed his little groups for massive and massless particles as two distinct mathematical devices. Indeed, Inonu and Wigner in 1953 initiated of the unification of these little groups by observing considering a flat plane tangent to a sphere, while the plane and sphere correspond to the E(2) and O(3) symmetries respectively [12]. This unification was completed in 1990 [2]. The issue is whether the E(2)-like little group can be obtained as a zero-mass limit of the O(3)-like little group
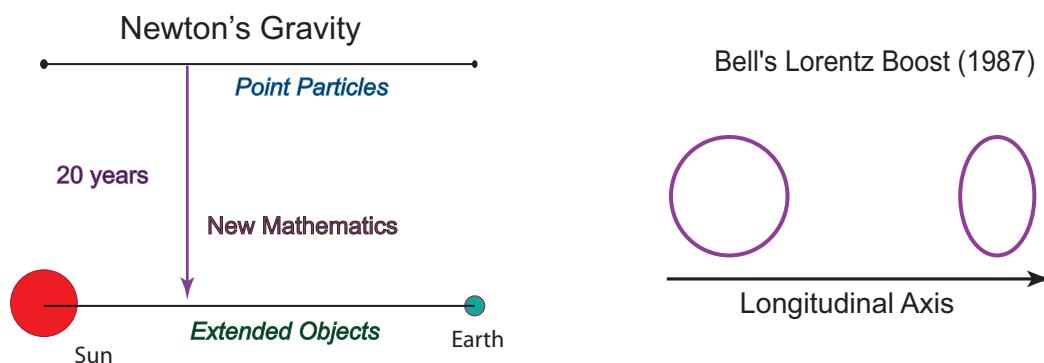
Figure 3: Newton's gravity law for point particles and extended objects. It took him 20 years to formulate the same law for extended objects. As for the classical picture of Lorentz contraction of the electron orbit in the hydrogen atom, it is expected that the longitudinal component becomes contracted while the transverse components are not affected. In the first edition of his book published in 1987, 60 years after 1927, John S. Bell included this picture of the orbit viewed by a moving observer [11]. While talking about quantum mechanics in his book, Bell overlooked the fact that the electron orbit in the hydrogen atom had been replaced by a standing wave in 1927. The question then is how standing waves look to moving observers.

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for massive particles. Another version of this limiting process is given in Sec. 5 of the present report.

As for the internal space-time symmetry of particles, let us go beck to Bohr and Einstein. Bohr was interested in the electron orbit of the hydrogen atom while Einstein was worrying about how things look to moving observers. They met occasionally before and after 1927 to discuss physics. Did they talk about how the stationary hydrogen atom would look to a moving observer? It they did, we do not know about it.

This problem is not unlike the case of Newton'a law of gravity. He worked out the inverse square law for two point particles. It took him 20 years to work out the same law for extended objects such and the sun and earth, as illustrated in Fig. 3.

In 1905, Einstein formulated his special relativity for point particles. It is for us to settle the issue of how the electron orbit of the hydrogen atom looks to moving observers. Indeed, the circle and ellipse as given in Fig. 3 have been used to illustrate this this relativistic effect. However, these figures do not take into account the fact that the electron orbit had been replaced by a standing wave. Indeed, we should learn how to Lorentz-boost standing waves.

Yes, we know how to construct standing waves for the hydrogen atom. Do we know how to Lorentz-boost this atom? The answer is No. However, we can replace it with the proton without changing quantum mechanics. Both the hydrogen atom and the proton
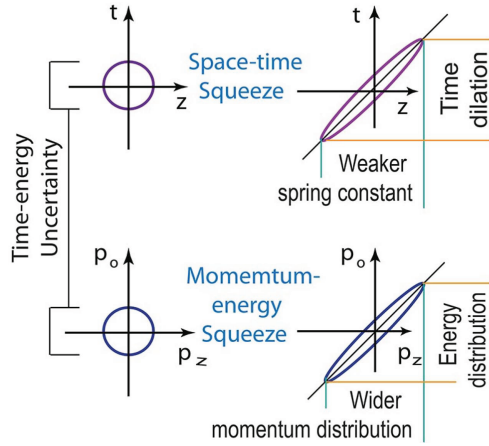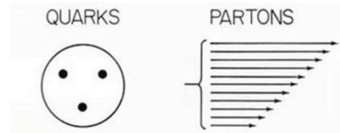
Figure 4: Gell-Mann's Quark model and Feynman's parton model as one Lorentz-covariant entity. The circle-ellipse diagram is from Ref. [10].

are quantum bound states, but the proton can be accelerated. While the Coulomb force is applicable to the hydrogen, the harmonic oscillator potential is used as the simplest binding force for the quark model [13]. We can switch the Coulomb wave functions with oscillator wave functions without changing quantum mechanics. This problem is illustrated in Fig. 4. Then it is possible to construct the oscillator wave functions as a representation of Wigner's little group [6, 14].

With those oscillator functions, it is possible to construct representations of Wigner's little group for the massive proton. It is then possible to Lorentz-boost the wave function to obtain the parton model satisfying all the peculiarities of the partons [15]. Indeed, this is possible thanks to Wigner's little groups that dictate the internal space-time symmetries of relativistic extended particles [2, 6, 14], as indicated in Table. 1.

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