So... $\Gamma^{\alpha}{}_{\nu\mu,\beta} + \Gamma^{\alpha}{}_{\nu\beta\mu} = \frac{1}{3}\left(R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\beta\nu}\right).$

Now project out symmetric part in $(\beta\nu)$

$$\Gamma^{\alpha}{}_{\mu(\nu,\beta)} + \Gamma^{\alpha}{}_{\nu\beta,\mu} = -\frac{1}{3} R^{\alpha}{}_{(\beta\nu)\mu}$$

$$\parallel$$

$$\frac{1}{3} R^{\alpha}{}_{(\beta\nu)\mu}$$

So $$\boxed{\Gamma^{\alpha}{}_{\nu\beta,\mu} = -\frac{2}{3} R^{\alpha}{}_{(\beta\nu)\mu}} \quad (\text{at } p)$$

Rename $\beta \leftrightarrow \mu$, and lower α index (which we do with impunity on Γ term since $g_{\alpha\beta,\mu}(p) = 0$).

$$\boxed{\Gamma_{\alpha\mu\nu,\beta} = -\frac{2}{3} R_{\alpha(\mu\nu)\beta}} \quad (\text{at } p)$$

Now solve for $g_{\mu\nu,\alpha\beta}$. Note in general $\boxed{g_{\mu\nu,\alpha} = 2\Gamma_{(\mu\nu)\alpha}}$

So $$g_{\mu\nu,\alpha\beta} = 2\Gamma_{(\mu\nu)\alpha,\beta}$$

$$= -\frac{2}{3}\left\{ R_{\mu(\nu\alpha)\beta} + R_{\nu(\mu\alpha)\beta} \right\}$$

$$= -\frac{1}{3}\left\{ \cancel{R_{\mu\nu\alpha\beta}} + R_{\mu\alpha\nu\beta} + \cancel{R_{\nu\mu\alpha\beta}} + R_{\nu\alpha\mu\beta} \right\}$$

$$\boxed{g_{\mu\nu,\alpha\beta} = +\frac{2}{3} R_{\mu(\alpha\beta)\nu}} \quad (\text{at } p). \qquad \text{QED.}$$