Riemann normal coordinates

$X^\alpha(\lambda, V^\gamma) = V^\alpha \lambda$: n-param. family of radial geodesics

Tangents: $\dot{X}^\alpha = V^\alpha$

Connecting vectors $C^\alpha_{(\gamma)} = \frac{\partial X^\alpha}{\partial V^\gamma} = \delta^\alpha_{(\gamma)} \lambda$, $\dot{C}^\alpha = \delta^\alpha_{(\gamma)} = \lambda^{-1} C^\alpha$

$$V^\mu \nabla_\mu (V^\nu \nabla_\nu C^\alpha) = - R^\alpha{}_{\beta\mu\nu} V^\beta V^\nu C^\mu$$

Find 1st order term in λ on both sides.

$\Gamma = O(\lambda)$, so Γ in the second derivative op must act on

$O(\lambda^0)$ term: $V^\nu \nabla_\nu C^\alpha = V^\nu \left(\frac{\partial}{\partial x^\nu} C^\alpha + \Gamma^\alpha_{\nu\beta} C^\beta \right) = \underbrace{\frac{1}{\lambda} C^\alpha}_{O(\lambda^0)} + \underbrace{V^\nu \Gamma^\alpha_{\nu\beta} C^\beta}_{O(\lambda^2)}$

Schematically, $(\partial + \Gamma)(\partial + \Gamma) C = \underbrace{\partial\partial C}_{\to 0} + \underbrace{\Gamma \partial C + \partial(\Gamma C)}_{O(\lambda^1)} + \underbrace{\Gamma\Gamma C}_{O(\lambda^3)}$

Note since $\Gamma(\lambda=0) = 0$, $\Gamma = \dot{\Gamma}\lambda + O(\lambda^2)$, so l.h.s. to $O(\lambda)$ is

$$V^\mu \dot{\Gamma}^\alpha_{\mu\beta} C^\beta + V^\nu \dot{\Gamma}^\alpha_{\nu\beta} C^\beta + V^\nu \dot{\Gamma}^\alpha_{\nu\beta} C^\beta = 3 \Gamma^\alpha{}_{\beta\mu,\nu} V^\mu V^\nu C^\beta$$

So $\Gamma^\alpha{}_{\beta\mu,\nu} V^\mu V^\nu \delta^\beta_{(\gamma)} = \frac{1}{3} R^\alpha{}_{\beta\mu\nu} V^\mu V^\beta \delta^\nu_{(\gamma)} + \ldots$

or, renaming $\nu \leftrightarrow \beta$ on l.h.s., and peeling off $\delta^\nu_{(\gamma)}$ and V's,

$$\boxed{\Gamma^\alpha{}_{\nu(\mu,\beta)} = \frac{1}{3} R^\alpha{}_{(\beta\mu)\nu}} \quad \text{(at } p\text{)}.$$