HW#3 —Phys776—Spring2003 Prof. Ted Jacobson
Due Monday, April 7, 2003, or earlier. Room 4115, (301)405-6020
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1. *Temporal size of a Schwarzschild black hole interior*

   Show that the longest timelike curve inside the horizon has a length πM. (Since extrema of the length are geodesics, any such curve is a geodesic. Thus to live the maximal time inside such a black hole you should refrain from firing your rockets. No matter which way you orient them you will only decrease your lifetime. This also means that in finding the time, you can assume the curve is a geodesic.)

2. *Hypersurface orthogonal vector fields*

   A *stationary* spacetime is one with a timelike Killing vector. If the timelike Killing vector is also "hypersurface orthogonal", the spacetime is called *static.*

   If a vector field V^a is hypersurface orthogonal, then there is some function S whose level sets are the hypersurfaces. Since V^a and $\nabla^a S$ are both orthogonal to all vectors tangent to the constant S hypersurfaces, they must be proportional, i.e. $V_a = f\nabla_a S$ for some function f.

   (a) Show that if V^a has this form then

   $$V_{[a}\nabla_b V_{c]} = 0. \tag{1}$$

   (The converse is *Frobenius' theorem.*)

   (b) Use (1) to show that, in Minkowski spacetime, the Killing field $(\partial/\partial t)+\Omega(\partial/\partial\phi)$ (Ω is constant) is *not* hypersurface orthogonal.

3. *Eddington-Finkelstein coordinates and Schwarzschild coordinates*

   In these two coordinate systems the spherically symmetric vacuum black hole line element takes the form

   $$\begin{aligned} ds^2 &= -(1-1/r)dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\phi^2) & (2)\\ &= -(1-1/r)dt^2 + (1-1/r)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), & (3)\end{aligned}$$

   in units with the Schwarzschild radius equal to one, $r_s = 2GM/c^2 = 1$.

   (a) Show that $v = t + r_*$, where r_* is the Regge-Wheeler "tortoise coordinate"

   $$r_* = r + \ln(r-1). \tag{4}$$

   You could of course verify this by just inserting the definition of v or t in (2) or (3) respectively, but that doesn't teach you how you would have found the "advanced time" or "ingoing Eddington-Finkelstein coordinate" v in the first place. Instead, note from the Schwarzschild line element that ingoing and outgoing radial light rays satisfy $dt^2 = dr^2/(1-1/r)^2$. Using this observation, show that the combination of t and r that is constant on ingoing radial light rays is v and use this path to relate the two line elements.

(b) Show that the line element can also be written in the "conformal" form

$$ds^2 = (1 - 1/r)(-dt^2 + dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$