HW#2 —Phys776—Spring2003
Due Monday, April 7, 2003, or earlier.

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1. *Hamilton's equations*

   As discussed in class (see Breno's notes from 2-26 and 3-3) phase space is a $2n$ dimensional manifold with a "symplectic structure", i.e. a closed non-degenerate 2-form Ω, and Hamilton's equations take the form $i_{X_H}\Omega = -dH$, where H is the Hamiltonian and X_H is the Hamiltonian vector field. Show that this is equivalent to the standard form of Hamilton's equations you'd find in Goldstein in terms of q^i and p_i, where these are coordinates on phase space such that $\Omega = dp_i \wedge dq^i$ (summation on i implicit).

   [*Background info*: If the phase space is the cotangent bundle T_*Q of a configuration space Q, then $\Omega = d\theta$, where θ is the canonical 1-form. In local coordinates q^i for Q, with induced coordinates p_i for the one-forms, we have $\theta = p_i dq^i$. (In fact even if the phase space is not a cotangent bundle, Darboux's theorem tells us that locally there exist coordinates for which $\Omega = dp_i \wedge dq^i$.)]

2. *Stokes' theorem*

   Show that the Stokes and divergence theorems in 3d vector calculus are special cases of the Stokes theorem for differential forms, $\int_{\mathcal{R}} d\omega = \int_{\partial\mathcal{R}} \omega$.

3. *Magneto-hydrodynamics*

   A common approximation for a plasma is a fluid with a definite four-velocity u at each point of spacetime. For a perfectly conducting plasma, the electric field vanishes in the rest frame of the plasma, i.e. $i_u F = 0$, were $F = dA$ is the electromagnetic field strength 2-form and A is the vector potential. This is called an "ideal plasma".

   (a) Show that under these assumptions the Lie derivative of F along the fluid flow vanishes, i.e. $\mathcal{L}_u F = 0$. This is the famous "frozen-in theorem" of plasma physics. (Note that it is a metric-independent statement, hence holds, for example, for a relativistic plasma in a rotating black hole spacetime.)

   (b) The frozen-in theorem gets its name from the fact that the magnetic flux is locked to the flow, in the sense that the flux through a loop is unchanged as the loop is carried along by the flow. Using the previous result show that this condition indeed holds .

   (c) The *magnetic helicity* density of a plasma is the 3-form $h = A \wedge F$, and the *helicity* in a spatial region V_3 is defined by $H = \int_{V_3} h$. In the rest of this problem you will show that under certain circumstances the helicity H is a conserved quantity.

      i. Show that for the electromagnetic field in an ideal plasma we have $dh = 0$. (*Hint*: This would not be true if spacetime had more than four dimensions.)

      ii. Using the previous result, Stokes' theorem tells us that $\oint_{\partial V_4} h = 0$ for any four-dimensional region V_4. Since this conclusion holds in any gauge the integral must be gauge invariant. Show this directly by making a gauge transformation $A \to A + d\lambda$.

iii. Suppose V_4 is the region swept out by a three-dimensional region V_3 as it is translated in time from t_1 to t_2. The boundary ∂V_4 then consists of the spacelike pieces $V_3(t_1)$ and $V_3(t_2)$, and the timelike piece consisting of ∂V_3 translated through time. The contributions from the spacelike boundaries give $H(t_2) - H(t_1)$, so we get a conservation of helicity in V_3 provided the integral over the timelike boundary vanishes. This is the case if the field is zero there, e.g. if the boundary is at infinity and the field falls off fast enough, but it can also be true in a finite region even if the field does not vanish. For example the region could be the interior of a tokamak. Show that the timelike boundary contribution vanishes if the normal component of the magnetic field and the parallel component of the electric field vanish at the boundary of the three volume ∂V_3. (For extra credit, remind me why these are reasonable boundary conditions if the boundary is a conductor!)

iv. Show that H is gauge invariant provided the boundary condition discussed in the previous part holds.