Complex impedance method for AC circuits

An alternating current (AC) circuit is a circuit driven by a voltage source (emf) that oscillates harmonically in time as $V = V_0 \cos \omega t$. In the physical regime where non-linear effects can be neglected, the response is linear. Thus, aside from transients, the current also oscillates harmonically, and takes the form $I = I_0 \cos(\omega t - \phi)$. The amplitude I_0 and phase ϕ are determined by the driving voltage and the nature of the circuit.

The amplitude of the current is linearly related to the amplitude of the voltage, but the phase angle is determined by a trigonometric equation. These relations can be very conveniently combined into a single linear relation by expressing the voltage and current as the real parts of complex quantities

$$\hat{V} = \hat{V}_0 \exp(i\omega t) \qquad and \qquad \hat{I} = \hat{I}_0 \exp(i\omega t), \tag{1}$$

with $\hat{V}_0 = V_0$ and $\hat{I}_0 = I_0 e^{-i\phi}$. We use the hat notation to indicate a quantity that is a complex number. The *impedance* Z is defined as the ratio of the complex voltage and current amplitudes:

$$Z = \frac{\hat{V}_0}{\hat{I}_0} = \frac{V_0}{I_0} e^{i\phi}. \tag{2}$$

(Since Z is almost always complex we don't bother to put a hat on it.) The complex voltage $\hat{V}$ and current $\hat{I}$ (1) thus obey the linear relation $\hat{V} = \hat{I} Z$, which is a complex generalization of Ohm's law, $V = IR$.

The impedance is most directly interpreted when written in polar form, $Z = |Z| e^{i\phi}$. The magnitude $|Z| = V_0/I_0$ is called the *reactance*, and it determines the real amplitude of the current given the real amplitude of the voltage. The phase ϕ of Z encodes the phase relation between voltage $V_0 \cos \omega t$ and current $I_0 \cos(\omega t - \phi)$.

Circuit elements

Each type of circuit element is characterized by its own impedance.

Resistor

The current in a resistor R is governed by Ohm's law, $V = IR$, so the impedance of a resistor is just the resistance,

$$Z_R = R. \tag{3}$$

The reality of Z_R expresses the fact that the current in a resistor is in phase with the voltage across it.

Inductor

For an inductor we have $V = LdI/dt$. Substituting the complex voltage and current (1) yields the relation $\hat{V} = (i\omega L)\hat{I}$, so the impedance of an inductor is given by

$$Z_L = i\omega L. \tag{4}$$

The impedance of an inductor differs from that of a resistor in two ways: it depends on frequency and it is an imaginary number.

The dependence of Z_L on the frequency arises from the fact that the voltage is proportional to the *derivative* of the current rather than the current itself. Note that at higher frequencies the impedance of an inductor is larger, so for a given current the voltage is larger. This reflects the fact that at higher frequencies the current changes more rapidly, so the magnetic flux through the inductor changes more rapidly, so the induced emf is greater. At lower frequencies on the contrary, the inductor behaves more like a short circuit, since it presents less opposition to a slowly varying current.

The fact that Z_L is imaginary reflects the fact that the current is $\pi/2$ out of phase with the voltage. The voltage is proportional to the derivative of the current, hence if the voltage oscillates as $\cos \omega t$ the current must oscillate as $\sin \omega t = \cos(\omega t - \pi/2)$. The voltage therefore leads the current by $\pi/2$. This is why the phase of the impedance is $\pi/2$: $Z_L = i\omega L = \omega L e^{i\pi/2}$.

Capacitor

The analysis for a capacitor is similar to that for an inductor. For a capacitor $V = Q/C$, hence $dV/dt = I/C$ (since $I = dQ/dt$). Substituting the complex voltage and current (1) thus yields $i\omega\hat{V} = \hat{I}/C$, or $\hat{V} = \hat{I}/i\omega C$. The impedance of a capacitor is thus given by

$$Z_C = 1/i\omega C. \tag{5}$$

Like for an inductor, the impedance of a capacitor depends on frequency and is an imaginary number. However, the dependence is inverted, since the voltage is proportional to the *anti-derivative* of the current rather than the derivative. At higher frequencies the impedance of a capacitor is smaller. This reflects the fact that the current reverses more quickly, so the capacitor has less time to fill with charge, so it behaves more like a short circuit. At lower frequencies, on the contrary, the impedance is greater since the charge builds up and capacitor behaves more like an open circuit.

The phase shift for a capacitor is opposite that for an inductor: the voltage lags the current by $\pi/2$, so the phase of the impedance is $-\pi/2$: $Z_C = 1/i\omega C = (1/\omega C)e^{-i\pi/2}$.

Combining impedances

The beauty of the complex impedance method is that the impedances add in series and in parallel exactly as do resistances. In the series case, $Z = Z_1 + Z_2$, and in the parallel case $1/Z = 1/Z_1 + 1/Z_2$. This means that any circuit can be reduced to a single equivalent circuit element, with a complex impedance that is neither purely real nor purely imaginary.

Power

As with a driven mechanical oscillator, it is useful to know how much power is absorbed by an AC circuit driven by an external potential $V = V_0 \cos \omega t$. The work done by the external potential in driving a charge q through a potential difference V is qV. Therefore the rate of doing work on the charges in a circuit, i.e. the power, is $P = V\,dq/dt = VI$. In an AC circuit, just as in a mechanical oscillator, the sign of this oscillating work is not in general constant in time. What is relevant is the *average* power $\langle P \rangle = \langle VI \rangle$.

Let us see how the average power depends on the impedance and the voltage. To compute this average we cannot use the complex quantities $\hat{V}$ and $\hat{I}$ since the product VI is not linear. We must first extract the real parts, then multiply and take the time average. The real current is given by $I = \mathrm{Re}[(V_0/Z)\exp i\omega t] = (V_0/|Z|)\cos(\omega t - \phi)$ where ϕ is the phase of Z. Since $\langle \cos \omega t \cos(\omega t - \phi) \rangle = (1/2)\cos\phi$, the time average of the power is

$$\langle P \rangle = (V_0^2/2|Z|)\cos\phi. \tag{6}$$

For a pure capacitance or pure inductance, Z is pure imaginary, so $\phi = \pm\pi/2$, so the average power is zero. That means that no energy is dissipated in those circuit elements. They *store* energy but they don't dissipate it. For a pure resistance $Z = R$ is real, so $\phi = 0$, so the average power is $\langle P \rangle = V_0^2/2R$. This may not immediately look like the usual relation for DC circuits, $P = V^2/R$, but it is in fact equivalent, since the average value of V^2 is just $V_0^2/2$.

Introducing the *root mean square* voltage

$$V_{rms} = \sqrt{\langle V^2 \rangle} = V_0/\sqrt{2}, \tag{7}$$

the average power (6) can be written as $\langle P \rangle = (V_{rms}^2/|Z|)\cos\phi$. The rms voltage and current are the quantities usually referred to for AC circuits, rather than the amplitudes themselves, which are a factor of $\sqrt{2}$ larger.

Generalizations

The method of complex impedance is applicable to any system whose response is linearly related to an input. Since almost all systems have a linear response near an equilibrium configuration, the method is almost universally applicable.