Chapter 5

Chiral Dynamics

5.1 What is spontaneous symmetry breaking?

Symmetries and their breakings are important part of modern physics. Spacetime symmetry and its supersymmetric extensions are the basis for building quantum field theories. Internal symmetries, such as isospin (proton and neutron, up and down quark symmetry), flavor, color etc., form the fundamental structure of the standard model. On the other hand, studying symmetry breakings is as interesting as studying symmetries themselves. As far as we know, there are three ways to break a symmetry: explicit breaking, spontaneous breaking, and finally anomalous breaking. In this part of the lectures we will concern ourselves with the first two types of breakings of the so-called chiral symmetry, the exact meaning of which will become clear later. We will come to the anomalous symmetry breaking towards the end of the course.

In quantum mechnics, a symmetry of a hamiltonian is usually reflected in its energy spectrum. For instance, the rotational symmetry of a three-dimensional system often leads to a $2\ell + 1$-fold degeneracy of the spectrum. This standard realization of a symmetry is called Wigner-Weyl mode. On the other hand, in the late 50's Nambu and Goldstone discovered a new way through which a symmetry of a system can manifest itself: spontaneous breaking of the symmetry. This realization of a symmetry is called Nambu-Goldstone mode.

To understand the Nambu-Goldstone realization of a symmetry, let us recall a related problem in statistical mechanics: second-order phase transitions. We have many examples of the second-order phase transitions in which a continuous change of order parameters happens. Consider a piece of magnetic material. Its hamiltonian is certainly rotationally symmetric and therefore normally one would expect its ground state wave function is also rotationally symmetric. This apparently is not the case below a certain critical temperature at which a spontaneous magnetization occurs. The magnitization vector points to a certain direction in space, and hence the rotational symmetry is lost. We say in this case that the rotational symmetry is spontaneously broken. Likewise, for a conductor below a certain temperature, the electromagnetic U(1) symmetry is spontaneously broken and the wavefunction of the Cooper pairs developes certain classical value.

A useful mathematical formulation of the SSB is the concept of the effective action. Let us introduce this first.

Consider a scalar field theory with lagrangian density $\mathcal{L}(\phi)$. We define the green's function functional or generating functional $Z(j)$ as

$$Z(j)=\sum_{i=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots d^4x_n j(x_1)\cdots j(x_n)G^{(n)}(x_1,\cdots,x_n) \tag{5.1}$$

where $G^{(n)}(x_1,\cdots,x_n)=\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$. In the path integral formulation, we have

$$Z(j)=\frac{\int[D\phi]e^{i\int d^4x(\mathcal{L}(x)+\phi(x)j(x))}}{\int[D\phi]e^{i\int d^4x\mathcal{L}(x)}}\ . \tag{5.2}$$

We define the connected green's function $G_c^{(n)}$ through

$$W(j)=\sum_{i=1}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots d^4x_n j(x_1)\cdots j(x_n)G_c^{(n)}(x_1,\cdots,x_n) \tag{5.3}$$

and $e^{iW(j)}=Z(j)$. In Feynman diagram expansion, the connected Green's functions contain diagrams with no disconnected parts. The physical significance of the $-W$ is that it equals to the time T times the ground state energy of the system $TE(J)$. This is the case because J is turned on gradually and the system reaches the ground state at $t=0$. The evolution of the state has a phase factor e^{-iET}. Define classical field $\phi(x)$ through

$$\phi(x)=\frac{\delta W(j)}{\delta j(x)}\ , \tag{5.4}$$

from which one can solve $j(x)$ as a functional of $\phi(x)$. Perform now the Legendre transformation ,

$$\Gamma(\phi)=\left(W-\int d^4xj(x)\phi(x)\right)|_{j=j(\phi)} \tag{5.5}$$

Then $\Gamma(\phi)$ is the generating functional for the one-particle irreducible Green's functions $\Gamma^{(n)}(x_1,\cdots,x_n)$,

$$\Gamma(\phi)=\sum_{n=1}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\Gamma^{(n)}(x_1,\cdots,x_n)\phi(x_1)\cdots\phi(x_n) \tag{5.6}$$

For example, $\Gamma^{(2)}(p)=iG_2(p)^{-1}=p^2-m^2-\Sigma(p)$. $\Gamma(\phi)$ is also called the quantum effective action for the following reason. Consider the minimum energy of the system under the constraint that the expectation value of the field $\hat\phi$ must be ϕ_c. The solution is that it is equal to $-\Gamma(\phi)/T$. This not only can be proved through the variational approach, and is also obvious from the Legendre transformation: The ϕ_c becomes the controlling variable in the thermodynamical study. In the tree level approxiamtion, $\Gamma(\phi)=\int d^4x\mathcal{L}(\phi)$. The expectation value of quantum field ϕ is clearly an extreme of Γ because,

$$j(x)=-\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\ . \tag{5.7}$$

Effective action can be computed through the shift of field in the lagrangian $\phi\to\phi+\phi_c$, and calculating the 1PI contribution to the effective W.

There are two popular usage of the effective action formalism: First, the effective action contains all the 1PI which are the target for renormalization study. The renormalization condition can easily expressed in terms of 1PI, like the mass of the particles and coupling constants. Moreover, the symmetry of these 1PI can be expressed in terms of the Ward-Takahashi identities which can be summarized in terms of a simple equation for the effective action. This equation can be used to prove the Goldstone theorem. Second, the effective action can be used as a thermodynamic function with natural variable ϕ_c which diagnoses the phase structure of the system. For instance, according to Colemann-Weinberg, the natural phase of the massless scalar electrodynamics is the Higgs phase in which the vector and scalar particles aquire mass through radiative corrections. Another use of the effective action is in cosmology.

The spontanous symmetry breaking happens only if there is a degeneracy in the vacuum. This degeneracy can arise from certain symmetry of the original lagrangian. Consider a symmetry transformation of fields,

$$\phi_i(x) \to \phi_i'(x) = \sum_j L_{ij}\phi_j(x) \ , \tag{5.8}$$

here we have assumed multiple fields with $i = 1, ..., n$. If the action and measure are both invariant, then the effective action is invariant under a similar transformation of the classical fields

$$\Gamma[\phi] = \Gamma[L\phi] \ . \tag{5.9}$$

As we mentioned before, the vaccum state is a solution $\bar{\phi}$ of $-\Gamma[\phi]$ at its minimum. If the solution is invariant $L\bar{\phi} = \bar{\phi}$, i.e. the vacuum is invariant under the symmetry transformation, the vacuum is unique. On the other hand, if $L\bar{\phi} \neq \bar{\phi}$, the solution is not. Then we have many degenerate vacua which are all physically equivalent. By choosing a particular $bar\phi$ as the true vaccum, we have a spontaneous symmetry breaking.

According to the above discussion, the key condition for SSB is there are multiple, equivalent vacua. Although it is easy to find ground state degeneracies in the classical systems, in quantum systems it is difficult to have multiple vacuum. For instance, in a potential with a double well, the ground state is a non-degenerate symmetrical state. In other words, the real vacuum is a linear combination of the various classical vacua. The same thing happens for a rotationally symmetric system in which the ground state has $J = 0$, i.e., all θ angles are equally probable.

There are special cases in quantum mechanics in which the ground state may be degenerate. For instace, in an atom with a ground state $J \neq 0$, the state can be prepared in the eigenstates of J^2 and J_z. However, there is no SSB because the states of different J_z are not equivalent vacua in the sense that they blong to the same Hilbert space and are easily connected through a transitions operators. Therefore, the spontaneous symmetry breaking happens only if the volume of the system is approaching infinity and the transition rate between the degenerate states goes to zero. In this case, it turns out that the vacuum states are not representations of the symmetry generators. Rather they are eigenstates of the conjugating coordinate operators and are superposition of states with symmetry quantum numbers. Any perturbation which causes the transition between different vacua have exponentially small matrix elements. On the other hand, the diagonal matrix elements of the perturbation is much larger than the off-diagonal matrix elements. In other words, the vacuum states are those with definite $\bar{\phi}$, or in the rotationally symmetric system, definite θ. So in the limit of infinit volume, the states with definite $\bar{\phi}$ become the exact vacua.

It can be shows that with local hamiltonian and operators, different vacua obey the *super-selection rule*. Assume the degenerate vacua are $|v_i\rangle$ and

$$\langle v_i | v_j \rangle = \delta_{ij} \tag{5.10}$$

By considering the matrix element of $\langle v_i|A(\vec{x})B(0)|v_j\rangle$ in the limit of $\vec{x}\to\infty$, it can be shown

$$\langle u_i|A(0)|u_j\rangle = \delta_{ij}a_i \ . \tag{5.11}$$

Therefore the local operators have no finite matrix elements between different vacuum states.

5.1.1 SSB and Space(-time) Dimensions

In a finite quantum mechanical system, there is no SSB. For discrete symmetry, such as Z_2 symmetry ($\sigma_i \to -\sigma_i$) in the Ising model, it cannot be broken in one-dimensional (0+1) system. This is known in 1938 to Peierls. But, it can be broken in two-dimensional (1+1) system. For example, the Onsagar solution contains a spontaneous magnetization for a two-dimensional Ising model.

For continuous symmetry, it cannot be spontaneously broken in two-dimensional system. This is called the Mermin-Wagner-Coleman theorem. For example, the classical Heisenberg model consists of interactions of spins living on a n-dimensional sphere. The system has $O(n)$ symmetry. This model has spontaneous symmetry breaking only in 3D. To see the MWC theorem, let's assume there is a SSB in 2D. Then we have massless Goldstone bosons. The correlation of these massless Goldstone bosons reads

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^2k}{2\pi}\theta(k^0)\delta(k^2)e^{ikx} = \int_0^\infty \frac{dk^1}{2\pi k^1} cos(k^1x^1)e^{ik^1x^0} \tag{5.12}$$

which is hopelessly infrared divergent. This strong fluctation will destroy any long-range order. In a two-dimensional classical Heisenberg model, an disordered phase has as much weight as an ordered one.

5.2 SSB of the continuous symmetry and Goldstone Theorem

In the case of the spontaneous breaking of a continuous symmetry, a theorem can be proved. The theorem says that the spectrum of physical particles must contain one particle of zero mass and spin for each broken symmetry generator. Those particles are called Goldstone bosons.

Consider an infinitesimal transformation

$$\phi_i \to \phi_i + i\epsilon_a(t^a\phi)_i \ . \tag{5.13}$$

The same transformation leaves the effective action invariant

$$\sum_{ij}\int d^4x \frac{\delta\Gamma}{\delta\phi_i(x)} t^a_{ij}\phi_j(x) = 0 \ . \tag{5.14}$$

If we look for a translationally-invariant solution such that ϕ is a constant, then $\Gamma = -V_3V(\phi)$, where V_3 is the 3-d volume and $V(\phi)$ is an effective potential in the usual sense. Then the spatial integral is trivial and we have

$$\sum_{ij}\frac{\partial V(\phi)}{\partial\phi_i}t^a_{ij}\phi_j = 0 \ , \tag{5.15}$$

This relation is true independent of ϕ. Differentiate the above equation with respect to ϕ_k and take $\phi = \bar{\phi}$ in a vacuum,

$$\frac{\partial^2 V(\phi)}{\partial_\ell\partial_i}(t^a\bar{\phi})_i = 0 \tag{5.16}$$

According to the definition of the effective potential, we have

$$\frac{\partial^2 V(\phi)}{\partial\phi_\ell \partial\phi_i} = \Delta_{\ell i}^{-1}(0) \ , \tag{5.17}$$

which is the inverse of the propagators at zero momentum. Then the equation

$$\Delta_{\ell i}(0)(t^a \bar{\phi})_i = 0 \tag{5.18}$$

is an eigenvalue equation. For a particular choice of ϕ, if there are n non-zero eigenvectors $t^a\bar{\phi}$, then there are n zero eigenvalues of $\Delta_{\ell i}^{-1}$ which correspond to n massless particles—Goldstone bosons.

Example. Consider $O(N)$ theory

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{2}M^2\phi_i\phi_i - \frac{g}{4}(\phi_i\phi_i)^2 \tag{5.19}$$

In the tree approximation $\Gamma = V_3\mathcal{L}$, we have

$$V = \frac{1}{2}M^2\phi_i\phi_i + \frac{g}{4}(\phi_i\phi_i)^2 \tag{5.20}$$

If M^2 is negative, we have

$$\bar{\phi}_i\bar{\phi}_i = -M^2/g \tag{5.21}$$

We can choose a solution as $\bar{\phi}_i = (0, ..., 0, \sqrt{M^2/g})$. There are $n-1$ generators which do not annihilate this state, therefore there n-1 goldstone bosons. The solution still has O(N-1) symmetry. The mass matrix is

$$\begin{aligned} M_{ij}^2 &= \frac{\partial^2 V(\phi)}{\partial\phi_i\partial\phi_j}|_{\phi=\bar{\phi}} \\ &= 2g\bar{\phi}_i\bar{\phi}_j = (0, ..., 0, 2|M^2|) \end{aligned} \tag{5.22}$$

Thus the last particle now has mass $\sqrt{2}M$.

There are $N(N-1)/2$ generators for $O(N)$ group. After SSB, $N-1$ generators no longer annihilate the vacuum. But the remainder $(N-1)(N-2)/2$ does. So the system still has $O(N-1)$ symmetry. The $N-1$ broken generators yield $N-1$ Goldstone bosons.

Consider a symmety current J_a^μ corresponding to a particular broken generator t_a which has an associate Goldstone boson a. J^a acting on the vacuum will not annihilate it, rather it will create a Goldstone boson state

$$\langle B_a|J_a^\mu(0)|0\rangle = iF_a p^\mu \tag{5.23}$$

where F_a is called the decay constant. It turns out that F_a is related to the vacuum expectation value (VEV) of ϕ and is an important parameter which characterizes the spontaneous symmetry breaking. More generally, we have

$$\langle B_b|J_a^\mu(0)|0\rangle = iF_{ab}p^\mu \tag{5.24}$$

The interaction between the zero-momentum Goldstone bosons can be deduced from the effective action immediately. Expand the effective action as a Taylor series in $\phi_i - \bar{\phi}_i = \sum_a Z_{ai}\pi_a + ...$ and using the result that

$$Z_{ai} = \sum_b F_{ab}^{-1}(it_b\bar{\phi})_i \tag{5.25}$$

we have the effective classical hamiltonian

$$H_{\rm eff} = \sum_N \frac{1}{N!} g_{a_1 a_2 \ldots a_N} \pi_{a_1} \cdots \pi_{a_N} \tag{5.26}$$

where

$$g_{a_1 a_2 \ldots a_N} = \sum Z_{a_1 i_1} \cdots Z_{a_N i_N} \frac{\partial^N V(\phi)}{\partial \phi_{i_1} \cdots \phi_{i_N}} \tag{5.27}$$

From equations derived earlier, it is easy to see that the amplitude for a zero-momentum Goldstone boson disappearing into the vacuum is zero. The amplitude for a zero-momentum goldstone boson to make transition to another boson is zero. Finally, the amplitude for three massless Goldstone bosons to make transtion is zero. This is in fact true to all orders.

Let us consider now the interactions of Goldstone bosons with other massive particles. The following approach assumes exact symmetry. To calculate the process of $\alpha \to \beta + B_a$, we start from the matrix element with the corresponding conserved current

$$\langle \beta | J_a^\mu | \alpha \rangle \ . \tag{5.28}$$

The current supports a momentum transfer $q = p_\alpha - p_\beta$. Clearly the most important contribution to matrix element comes from the Goldstone boson pole which has the following structure

$$\frac{iFq^\mu M_{\beta B,\alpha}}{q^2} \tag{5.29}$$

where we have calculated the Goldstone boson diagram with a pole $1/q^2$ and M is a matrix element of our interest. There is also *regular* contribution to the current matrix element without the Goldstone bosons $N^\mu_{J\beta,\alpha}$. The current conservation require that

$$M_{\beta+B,\alpha} = \frac{i}{F} q_\mu N^\mu_{\beta+J,\alpha} \ . \tag{5.30}$$

This is a form of Ward identity. If N^μ has no pole, then the process of emitting a Goldstone boson vanishes as $q \to 0$. This is called the Adler zero.

The most important contribution in the regular term comes from the Feynman diagrams in which J acting on the external line. In this case, there is a heavy-particle pole which enhance the contribution. The pole contribution can often be calculated or extracted from experimental data, from example, the nucleon pole contribution is related to the neutron beta decay constant g_A. Knowing g_A, we can calculate the meson-nucleon interaction as we shall do in the next section.

The above result can also be derived from a theory with explicit breaking of the symmetry. This approach is called PCAC. In this case, the masses of the Goldstone bosons are not exactly zero, but finite. They are called pseudo-Goldstone bosons. Let us consider the SSB of an approximate symmetry.

In this case, the vacuum is no longer degenerate, and strictly speaking, there is no spontaneous symmetry breaking. This is very much like a magnet in an external magnetic field (first order phase transition). In the following we would like to find the constraint on the vacuum from the symmetry breaking effects; we also want to derive the masses of the pseudo-Goldstone bosons.

Now the effective potential has two terms $V(\phi) = V_0(\phi) + V_1(\phi)$. The real solution is $\phi = \phi_0 + \phi_1$ which is no longer degenerate. The condition on ϕ_0 and ϕ_1 is contained in the expanded version of $\partial V(\phi)/\partial\phi_i|_{\phi=\bar{\phi}} = 0$

$$\frac{\partial^2 V_0}{\partial\phi_i\partial\phi_j}|_{\phi=\phi_0}\phi_{1j} + \frac{\partial V_1}{\partial\phi_i} = 0 \tag{5.31}$$

Using the equation we found early, we have

$$(t_a\phi_0)_i\frac{\partial V_1(\phi_0)}{\partial\phi_i} = 0 \tag{5.32}$$

The above equations says if we do not start with a proper ϕ_0, ϕ_1 cannot be regarded as a perturbation. This is also called the vacuum alignment condition which forces the direction of the symmetry breaking by the vacuum into some sort of alignment with the symmetry breaking term in the hamiltonian.

The mass matrix is

$$M^2_{ab} = \sum_{ij} Z_{ai}Z_{bj}\frac{\partial^2 V}{\partial\phi_i\partial\phi_j} \tag{5.33}$$

which vanish to the zeroth order. To the frist order, we find

$$M^2_{ab} = -\sum_{cd} F^{-1}_{ac}F^{-1}_{bd}\langle 0|[T_a,[T_b,H_1]]|0\rangle \tag{5.34}$$

where T_a is the quantum generator of the symmetry group.

5.3 pion as goldstone boson, PCAC

One of the most interesting examples of SSB is exhibited by fundamental strong interactions: quantum chromodynamics. Consider the QCD lagrangian. The only parameters with mass dimension are quark masses. For ordinary matter, we just consider up and down quark flavors. The QCD scale Λ_{QCD} is about 200 MeV, which is much larger than the up and down quark masses (5 to 9 MeV). Therefore, to a good approximation, we can negelect the quark masses in the QCD lagrangian. Then the QCD lagrangian has the U(2)×U(2) *chiral* symmetry.

Recall the chiral projection operators $P_L = (1-\gamma_5)/2$ and $P_R = (1+\gamma_5)/2$, where γ_5 is diag (-1,1), which project out the left-handed and right-handed quark fields,

$$\psi_{L,R} = P_{L,R}\psi\ . \tag{5.35}$$

Then the QCD lagrangian we can be written in terms of

$$\mathcal{L} = \overline{\psi}_L(i\not{D})\psi_L + \overline{\psi}_R(i\not{D})\psi_R - \frac{1}{4}F^{\mu\nu a}F_{\mu\nu a}\ . \tag{5.36}$$

where $\psi = (u,d)$ is a column vector in the flavor space. The lagrangian is invariant under the following transformations

$$\begin{aligned}\psi_L &\to U_L\psi_L\ ,\\ \psi_R &\to U_R\psi_R\ ,\end{aligned} \tag{5.37}$$

where $U_{L,R}$ are unitary matrices in the two-dimensional flavor space. Since U(2)=U(1)× SU(2), we have two U(1) symmtries. From now on, we focus on the two SU(2) symmetries, leaving the U(1) symmetries to later discussion.

According to Noether's theorem, the $\mathrm{SU}_L(2)\times\mathrm{SU}_R(2)$ chiral symmetry leads to the the following conserved currents ,

$$j^\mu_{L,R} = \bar\psi_{L,R} t^a \gamma^\mu \psi_{L,R} \ , \tag{5.38}$$

where $t^a = \tau^a/2$ and τ^a is the usual Pauli matrices. We have the vector and axial vector currents from the linear combinations,

$$\begin{aligned} j_V^{a\mu} &= \bar\psi t^a\gamma^\mu\psi = j^\mu_L + j^\mu_R \\ j_A^{b\mu} &= \bar\psi t^a\gamma^\mu\gamma_5\psi = j^\mu_R - j^\mu_L \ . \end{aligned} \tag{5.39}$$

From the above currents, we can define the charges Q_a and Q_{a5} in the usual way. And it is easy to see that the charges obey the following algebra:

$$[Q_a, Q_b] = i\epsilon_{abc}Q_c; \quad [Q_{5a}, Q_b] = i\epsilon_{abc}Q_{5c}; \quad [Q_{5a}, Q_{5b}] = i\epsilon_{abc}Q_c \ . \tag{5.40}$$

From the above, we find that Q_a forms a subgroup of the chiral symmetry group and is called the isospin group. From the experimental hadron spectrum, we find that the isospin subgroup is realized in Wigner-Weyl mode. For instance, the pion comes in with three charge states and near degenerate mass. The proton and neutron also have nearly degenerate mass. However, the spectrum does not show the full chiral symmetry. For instance, the three pion states do not form an irreducible reps of the chiral group. Together a scalar particle σ, they form (1/2,1/2) reps. Therefore, if the chiral symmetry is realized fully in Wigner-Weyl mode, there must be a scalar particle with the same mass as the pion. We do not see such a particle in Nature.

Thus, the chiral group $SU_L(2)\times SU(2)_R$ must break spontaneously to the isospin subgroup $SU(2)$. Thus the QCD vacuum $|0\rangle$ satisfies

$$Q_a|0\rangle = 0, \quad Q_{5a}|0\rangle \neq 0 \ . \tag{5.41}$$

According to Goldstone's theorem, there are three massless spin-0 pseudo-scalar bosons. They are pseudoscalars because Q_{5a} changes sign under parity transformation.

Of course, in the real world, we don't have massless pseudoscalars. We have pions. The pion masses are indeed much smaller than a typical hadron mass. For instance, the rho meson has mass 770 MeV. The nucleon mass is 940 MeV. And the pion mass is 140 MeV. The pions are called pseudo-Goldstone bosons because the chiral symmetry is not exact. It is broken by the finite up and down quark masses.

$$H_1 = m_u\bar uu + m_d\bar dd \ . \tag{5.42}$$

If we write u in terms of left and right-handed fields, we have

$$H_1 = m_u(\bar u_L u_R + \bar u_R u_L) + m_d(\bar d_L d_R + \bar d_R d_L) \ . \tag{5.43}$$

Therefore the left and right-handed fields are now coupled through the mass terms. The mass operator transforms as the components of (1/2,1/2) representations of the chiral group.

Using the relation we found earlier, we can calculate the pion mass,

$$m_\pi^2 = -(m_u+m_d)\langle 0|\bar uu + \bar dd|0\rangle/f_\pi^2 \tag{5.44}$$

where $\langle 0|\bar{u}u + \bar{d}d|0\rangle$ is the chiral condensate. Since $\bar{u}u$ is a part of the representation $(1/2, 1/2)$, it vacuum expectation value vanishes ordinarily because of the chiral symmetry. However, it has a vacuum expectation value because of the vacuum is no longer chirally symmetric (chiral singlet). In fact, the vacuum contains all (k, k) type of representations because the vacuum has zero isospin. Any chiral tensor of type (k, k) has non-zero vacuum expectation value.

The pion decay constant f_π is defined from

$$\langle 0|j_a^\mu(x)|\pi_b\rangle = ip^\mu \delta_{ab} f_\pi e^{-ip\cdot x} \ . \tag{5.45}$$

It can be measured from the semi-leptonic weak decay $\pi^+ \to \mu^+\nu_\mu$ rate

$$\Gamma = \frac{G_F^2 m_\mu^2 f_\pi^2 (m_\pi^2 - m_\mu^2)^2}{4\pi m_\pi^3} \cos^2\theta_C \ . \tag{5.46}$$

From the experimental data, one finds,

$$f_\pi = 93 \text{ MeV} \ . \tag{5.47}$$

As we shall see, f_π normaly appears with a factor of 4π and $4\pi f_\pi$ is about 1 GeV, the hadron mass scale.

Let us discuss the interaction between pion and nucleon systems. We first focus on the pion-nucleon interaction. To do this we start with the following nucleon matrix element

$$\langle p'|j_{5a}^\mu|p\rangle = \overline{U}(p')t_a[g_A(q^2)\gamma^\mu\gamma_5 + g_p(q^2)q^\mu\gamma_5]U(p) \ , \tag{5.48}$$

where $q = p - p'$ and U's are the on-shell Dirac spinors of the nucleon states. Multiplying q^μ to both sides of the equation and using current conservation and Direc equation $(\not{p} - M)U(p) = 0$, we have

$$-2Mg_A(q^2) + q^2 g_P(q^2) = 0 \ . \tag{5.49}$$

$g_A(q^2)$ in the limit of $q^2 \to 0$ is just the neutron decay constant (the axial current is part of the weak interaction current) and has been measured accurately

$$g_A(0) = 1.257 \ . \tag{5.50}$$

Thus according to the above equation $g_P(q)$ must have a pole in $1/q^2$. This pole corresponds to the intermediate massless pion contribution to the interaction between the the axial current and the nucleon. If we introduce the pion-nucleon interaction vertex $g_{\pi NN}\bar{N}i\gamma_5\tau^a N\pi_a$, the contribution to the axial current matrix element is

$$i^2 g_{\pi NN}\overline{U}\gamma_5\tau^a U\left(\frac{i}{q^2}\right)(if_\pi q^\mu) \ . \tag{5.51}$$

In the limit of $q^2 \to 0$, we find the following celebrated Goldberger-Treiman relation

$$g_A(0)M = g_{\pi NN} f_\pi \ . \tag{5.52}$$

Using $g_{\pi NN}$ from experimental data ($g_{\pi NN}^2/4\pi = 14.6$), we find that the above relation is obeyed at better than 10% level.

According to the recipe derived from the previous section, we calculate the interactions between the soft pion and the nucleon system as follows. First use a vertex iq^μ/f_π connecting the Goldstone boson to the axial current. Then the non-singular part of the axial current interaction with the nucleon is approximated through the $g_A\gamma^\mu\gamma_5$ vertex. This yields the effective pion-nucleon interaction vertex $i\not{q}\gamma_5/f_\pi$. This is a peudo-vector interaction.

Another way to study the interactions among the pions and with other particles is through what is called the PCAC (partially-conserved axial-vector current), in which we assume there is a small explicit symmetry breaking through nonvanishing quark masses. Applying the derivative operator to the current matrix between the vacuum and the pion, we have

$$\langle 0|\partial_\mu j^\mu_a|\pi_b\rangle = m_\pi^2\delta_{ab}f_\pi \ . \tag{5.53}$$

The right-hand side is proportional to the pion mass squared. This motivates the assumption that

$$\partial^\mu j^a_\mu = m_\pi^2 f_\pi \pi^a \ , \tag{5.54}$$

where π^a is a pion interpolating field. Of course, the above relation is in some sense empty because any pseudo-scalar operator can be used as an interpolating field for pion. The content of the PCAC is that axial current at zero momentum transfer (this is the place where we know how to calculate the matrix element) is dominated by the pion contribution at $q^2 = m_\pi^2$. In other words, the variation of the matrix elements of the axial current from $q^2 = 0$ to m_π^2 is smooth. In fact, we can derive the Goldberger-Treiman relation using PCAC and find now one has to use $g_A(0)$ instead of $g_A(m_\pi^2)$. The content of PCAC is that the variation of this small. Therefore, when the pion energy is small, we can calculate using PCAC.

PCAC can be used to study the multi-pion interactions. For instance, consider the amplitude

$$T^{ab}_{\mu\nu} = \int d^4x e^{iqx}\langle H(p_2)|TA^a_\mu(x)A^b_\nu(0)|H(p_1)\rangle \tag{5.55}$$

Applying differentical operators to the above quantity, we derive a Ward indentity. Using PCAC, one can calculate the pion-nucleon scattering amplitude at low-energy. However, it turns out that it is much easier to get the predictions using the low-energy effective theory.

5.4 the linear σ model

Many of the essential physics exhibited in spontaneous breaking of the chiral symmetry can be illustrated by a simple phenomenological model. This is very similar to the Ginsburg-Landau theory for second-order phase transitions. This model is first introduced by Gell-Mann and Levy, and is called the linear σ model. The lagrangian is,

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_S + c\sigma \ , \\
\mathcal{L}_S &= \bar\psi[i\not{\partial} + g(\sigma + i\vec\pi\cdot\tau\gamma_5)]\psi + \frac{1}{2}[(\partial\vec\pi)^2 + (\partial\sigma)^2] \\
&\quad - \frac{\mu^2}{2}(\sigma^2+\vec\pi^2) - \frac{\lambda}{4}(\sigma^2+\vec\pi^2)^2 \ ,
\end{aligned} \tag{5.56}$$

where $(\sigma,\vec\pi)$ forms a (1/2,1/2) representation of the chiral group and ψ, the nucleon (proton and neutron) field, form the (1/2,0)+(0,1/2) representation. In the absence of the symmetry breaking term $c\sigma$, the lagrangian is clearly symmetric under the chiral $SU_L(2)\times SU_R(2)$, and the corresponding vector and axial vector current is

$$
\begin{aligned}
j_a^\mu &= \bar\psi\gamma^\mu\frac{\tau^a}{2}\psi + \vec\pi\times\partial^\mu\vec\pi \\
j_{5a}^\mu &= \bar\psi\gamma^\mu\gamma_5\frac{\tau^a}{2}\psi + (\sigma\partial^\mu\pi^a - \pi^a\partial^\mu\sigma)
\end{aligned}
\tag{5.57}
$$

After introducing the symmetry breaking term, the axial vector current is no longer conserved. We have instead

$$\partial^\mu A_\mu^a = -c\pi^a \tag{5.58}$$

according to the equation of motion. The above has the form of PCAC.

When $\mu^2 < 0$, the spontaneous symmetry breaking happens. The potential has its minimum not at $\pi^a = \sigma = 0$ but at $\pi^2 + \sigma^2 = v^2$, where $v^2 = -\mu^2/\lambda$. Thus, the shape of the potential is a Mexican hat. There are infinite many degenerate minima. We need to choose a particular direction as our vacuum state. If we want to keep the isospin group intact, we take

$$\langle\sigma\rangle = v\ . \tag{5.59}$$

The pion excitation corresponds to the motion along the minima and therefore has zero energy unless the wavelength is finite. The σ mass corresponds to the curvature in the σ direction and is $2\lambda v^2$. The nucleon also get its mass from spontaneous symmetry breaking and is $-gv$. From the PCAC, we find that $f_\pi = -v$.

When the symmetry breaking term is introduced, the Mexican hat is tilted. In this case, the minimum of the potential is unique and the pion excitations do have mass.

5.5 effective field theory: Chiral Perturbation theory with pions

Current algebra and Ward identity approach were popular in the 60's for calculating Goldstone boson interactions. However, they are tedious. In 1967, Weinberg used the nonlinearly-transformed effective lagrangian to study the Goldstone boson interactions. This is the precursor of effective field theory approach which is popular today.

The key observation is that when the Goldstone boson energy is small, the coupling is weak. Therefore their interactions must be calculable in perturbation theory. However, in the strong interactions, we also have the usual QCD or hadron (rho meson or nucleon) mass scale. The physics at these two different scales have to be separated before one can apply chiral perturbation theory. The physics at QCD or hadron mass scale can be parametrized in terms of various low-enegy constants which can be determined from experimental data.

Through a particular model, we demonstrate the separation of physics through nonlinear transformations. We first perform a symmetry transformation at every point of the spacetime to get rid of the Goldstone boson degrees of freedom. We then re-introduce them through the spacetime-dependent symmetry transformation. When the Goldstone-boson fields are constant, the transformation is the usual chiral tranformation; and the Goldstone boson fields disappear. Therefore, in the new lagrangian, the Goldstone boson interaction must have derivative-type interactions.

Consider the linear sigma model. Let us introduce a (1/2,1/2) 2×2 matrix

$$U = \sigma + i\vec\pi\cdot\tau \tag{5.60}$$

Under the chiral transformation, we have

$$U \to U_L U U_R^\dagger \tag{5.61}$$

We can write the linear sigma model as

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] - \frac{\mu^2}{4}\mathrm{Tr}[UU^\dagger] - \frac{\lambda}{16}\left(\mathrm{Tr}[UU^\dagger]\right)^2 \tag{5.62}$$

We parametrize the chiral transformation in the following form

$$\begin{aligned} U_L &= e^{i(\theta_L^a \tau^a/2)}, \quad \theta_L^a = \theta_V^a + \theta_A^a \\ U_R &= e^{i(\theta_R^a \tau^a/2)}, \quad \theta_R^a = \theta_V^a - \theta_A^a \ . \end{aligned} \tag{5.63}$$

If we rotate away the Goldstone bosons, we have $U = \sqrt{\pi^2 + \sigma^2}$. We reintroduce back the goldstone boson by parametrizing the U including the axial transformation parameters,

$$U = \sigma e^{i\vec{\pi}^a(x)\cdot\tau^a/f_\pi} \tag{5.64}$$

where $\pi^a = f_\pi \theta_A^a$ is now the Goldstone boson field. For the convenience, we call the exponential factor Σ.

Now substituting $U = \sigma\Sigma$ into the original lagrangian, we get,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{4}\sigma^2\mathrm{Tr}[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger] - \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4 \ . \tag{5.65}$$

Now the Goldstone boson fields contain derivatives and therefore the above lagrangian will produce appropriate Goldstone boson interactions. Since the σ particle has a typical hadronic mass, its effects can be integrated out completely and the σ is then replaced by its expectation value. Therefore, the effective intereaction lagrangian for pion is just

$$\mathcal{L}_{\pi\pi}^{(2)} = \frac{f^2}{4}\mathrm{Tr}\left[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right] \tag{5.66}$$

The above lagrangian is sometimes called the nonlinear sigma model. One important point is that the above low-energy lagrangian is independent of any specific model that one starts with. The difference between models are the high momentum scale physics which we don't know how to capture any way. In this view, one shall be able to get the above effective low-energy lagrangian without any specific theory.

In fact at low-energy, the only object we can work with is Σ and its derivatives. Thus we can write done the interactions with increasing order of complication. For instance terms with four derivatives are

$$\mathcal{L}_{\pi\pi}^{(4)} = L_1\left(\mathrm{Tr}[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger]\right)^2 + L_2\mathrm{Tr}[\partial_\mu\Sigma\partial_\nu\Sigma^\dagger]\mathrm{Tr}[\partial^\mu\Sigma\partial^\nu\Sigma^\dagger] + L_3\mathrm{Tr}\left[\partial^\mu\Sigma\partial_\mu\Sigma^\dagger\partial^\nu\Sigma\partial_\nu\Sigma^\dagger\right] + ... \tag{5.67}$$

If all momentum are order of Q which is much smaller than the hadron mass scale, we can count the power of Q from a particular Feymnan diagrams. Consider an arbitrary graph with V_i number of vertices of type i which has d_i derivatives, with L loops and with I internal pion lines, the Q power is just

$$\nu = \sum_i V_i d_i - 2I + 4L \tag{5.68}$$

Using $L = I - \sum_i V_i + 1$, we have

$$\nu = \sum_i V_i(d_i - 2) + 2L + 2 \ . \tag{5.69}$$

Therefore the lowest power of Q in any pion process is 2.

We can use the above leading order lagrangian to calculate the interactions between the pions. Expand in $1/f_\pi$ to to the second order, we have

$$\mathcal{L}^{(2)}_{\pi\pi} = \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{6f_\pi^2}[(\partial_\mu \vec{\pi}\cdot\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi})^2] + ... \tag{5.70}$$

The second term can be used to calculate the S-matrix element between pion scattering. Assume the incoming pions with momenta p_A and p_B and isospin indices a and b and the outgoing pions with momenta p_C and p_D and isospin indices c and d. We have the following leading-order invariant amplitude ($S = 1 - iM$),

$$\mathcal{M} = -f_\pi^{-2}\left(\delta_{ab}\delta_{cd}s + \delta_{ac}\delta_{bd}t + \delta_{ad}\delta_{bc}u\right) \tag{5.71}$$

where $s = (p_A + p_b)^2$, $t = (p_A - p_C)^2$ and $u = (p_A - p_D)^2$ are called Mandelstam variables.

5.5.1 Scalar and Pseudoscalar Sources

We can include the quark mass effects at this order. The quark mass term transforms like $(1/2, 1/2)$ under chiral transformations. In general, let us introduce s and p source in the QCD lagrangian

$$\begin{aligned}\mathcal{L}_{sp} &= -\bar{\psi}s(x)\psi + \bar{\psi}i\gamma_5 p(x)\psi \\ &= -\bar{\psi}_R(s+ip)\psi_L - \bar{\psi}_L(s-ip)\psi_R\end{aligned} \tag{5.72}$$

Call $s - ip = \chi$ and $s + ip = \chi^\dagger$. Then the interaction is invariant if

$$\chi \to L\chi R; \quad \chi^\dagger \to R\chi^\dagger L^\dagger \tag{5.73}$$

Without the p source, $\chi \sim \chi^\dagger \sim s \sim m_q$, which counts as second order in momentum. The effective lagrangian then contain χ as a $\mathcal{O}(p^2)$ external source. The lowest order is

$$\mathcal{L}^{(2m)}_{\pi\pi} = B\mathrm{Tr}(\Sigma\chi^\dagger + \Sigma^\dagger\chi) \ . \tag{5.74}$$

When expanded to the leading order, the above gives the pion mass contribution if $B = f_\pi^2/4$ and $\chi = m_\pi^2$. The next-order contribution is

$$\frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2 \ , \tag{5.75}$$

which contributes to the π scattering as $-m_\pi^2/(3f_\pi^2)(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})$. The Q counting index now becomes $\nu = \sum_i V_i(d_i + 2m_i - 2) + 2L + 2$ where m_i is the number of insertions of quark masses. Combining the above contribution, the pion scattering invariant amplitude becomes

$$\mathcal{M} = -f_\pi^{-2}\left(\delta_{ab}\delta_{cd}(s - m_\pi^2) + \delta_{ac}\delta_{bd}(t - m_\pi^2) + \delta_{ad}\delta_{bc}(u - m_\pi^2)\right) \ . \tag{5.76}$$

At the threshold where $s = 4m_\pi^2$, $t = u = 0$, we have

$$\mathcal{M} = -m_\pi^2 f_\pi^{-2}[3\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}] \ . \tag{5.77}$$

The scattering amplitude $f = -\mathcal{M}/8\pi\sqrt{s}$ which is the scattering length a at low-energy (weinberg's definition, differing from Landau's by a minus sign). The scattering length in the $T = 0$ channel is $a_0 = 7m_\pi/32\pi f_\pi^2 = 0.16m_\pi^{-1}$ because of the wave function $\delta_{ab}\delta_{cd}/3$, and in the $T = 2 channel$ $a_2 = -2m_\pi/32\pi f_\pi^2 = -0.046m_\pi^{-1}$ with wave function $(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - 2/3\delta_{ab}\delta_{cd})/2$. The extra factor of 1/2 comes from the identity of two bosons. [The positive scattering length corresponds to repulsive interaction and negative one to attractive intereaction.] The above numbers can be compared to the experimental data $a_0 = .26 \pm 0.5$ and $a_2 = -0.028 \pm 0.012$ from $\pi + N \to 2\pi + N$ and $K \to 2\pi + e + \nu$.

To the next-order in pion momenta, we can calculate the one-loop contribution from $L_{\pi\pi}^{(2)}$. The contribution is divergent. The divergences can be cancelled by the high-order counter terms. The result is

$$\begin{aligned}\mathcal{M} \quad &= -\frac{\delta_{ab}\delta_{cd}}{16f_\pi^4}\left[\frac{-1}{2\pi^2}s^2\ln\left(\frac{s}{\mu^2}\right) - \frac{1}{12\pi^2}(u^2 - s^2 + 3t^2)\ln\left(\frac{-t}{\mu^2}\right)\right.\\ &\left.-\frac{1}{12\pi^2}(t^2 - s^2 + 3u^2)\ln\left(\frac{-u}{\mu^2}\right) - \frac{1}{2}c_1 s^2 - \frac{1}{4}c_2(t^2 + u^2)\right] + \text{crossing}\end{aligned} \tag{5.78}$$

where c_1 and c_2 are constants which must be determined from experimental data. In fact, there are also pion mass contribution at this order which we will not go into.

The p^4-order mass term include the following

$$\begin{aligned}&L_4\mathrm{Tr}(D^\mu\Sigma^\dagger D_\mu\Sigma)\mathrm{Tr}(\chi^\dagger\Sigma + \chi\Sigma^\dagger) + L_5\mathrm{Tr}(D^\mu\Sigma^\dagger D_\mu\Sigma)(\chi^\dagger\Sigma + \chi\Sigma^\dagger)\\ &+L_6(\mathrm{Tr}(\chi^\dagger\Sigma + \chi\Sigma^\dagger))^2 + L_7(\mathrm{Tr}(\chi^\dagger\Sigma - \chi\Sigma^\dagger))^2\\ &+L_8\mathrm{Tr}(\chi^\dagger\Sigma\chi^\dagger\Sigma + \chi\Sigma^\dagger\chi\Sigma^\dagger) + H_2\mathrm{Tr}(\chi^\dagger\chi)\end{aligned} \tag{5.79}$$

where H_2 is pointless because there is no meson depedence.

5.5.2 Electromagnetic and Axial Interactions

When there are electromagnetic and weak interactions with the Goldstone boson system, we need to construct a gauge theory in which the effective theory is gauge invariant under gauge transformations. Introduce the the following coupling the QCD lagrangian

$$\begin{aligned}\mathcal{L} \;&=\; \bar\psi(\gamma^\mu v_\mu(x) + \gamma^\mu\gamma_5 a_\mu(x))\psi\\ &=\; \bar\psi_L\gamma^\mu(v_\mu - a_\mu)\psi_L + \bar\psi_R\gamma^\mu(v_\mu + a_\mu)\psi_R\end{aligned} \tag{5.80}$$

If v_μ and a_μ are gauge fields, under gauge transformation, they must transform in the following way,

$$\begin{aligned}v_\mu - a_\mu &\to L(v_\mu - a_\mu)L^\dagger + iL\partial_\mu L^\dagger\\ v_\mu + a_\mu &\to R(v_\mu - a_\mu)R^\dagger + iR\partial_\mu R^\dagger\end{aligned} \tag{5.81}$$

The above equation means that these gauge fields have to appear together with Σ in the following form

$$D_\mu\Sigma = \partial_\mu\Sigma - i(v-a)_\mu\Sigma + i\Sigma(v+a)_\mu \tag{5.82}$$

Then all the partial derivatives will be replaced by the above covariant derivatives.

For example, consider the electromagnetic interaction of the pions. In this case, we replace $v_\mu = -ie(\tau_3/2 + 1/6)A_\mu$ where e is the charge of a proton and τ_3 is the isospin and 1/6 is the hypercharge. Then the partial derivative becomes,

$$D_\mu \Sigma = \partial_\mu \Sigma + ieA_\mu[\frac{\tau_3}{2}, \Sigma] \tag{5.83}$$

The resulting coupling with the isospin current of the pion in the leading order,

$$J^\mu = (\vec{\pi} \times \partial^\mu \vec{\pi}) \tag{5.84}$$

The leading order Feynman rule is

$$-i\epsilon^{a3b}(q_1 + q_2) \tag{5.85}$$

where a and q_1 are for the incoming pion, and b and q_2 for outgoing pion. The four-pion one-photon coupling is zero. The two-pion and two photon coupling is

$$2ie^2 g^{\mu\nu}(\delta^{ab} - \delta^{a3}\delta^{b3}) \tag{5.86}$$

Another example is the pion coupling with axial vector source. the leading vertex is

$$f_\pi \delta^{ab} k^\mu \tag{5.87}$$

And the coupling with three pions is

$$... \tag{5.88}$$

Moreover, at p^4 order, we have additional terms from guage fields

$$\begin{aligned} & L_9(-i\mathrm{Tr}(F^L_{\mu\nu} D^\mu \Sigma D^\nu \Sigma^\dagger) - i\mathrm{Tr}(F^R_{\mu\nu} D^\mu \Sigma^\dagger D^\nu \Sigma)) \\ & L_{10}\mathrm{Tr}(\Sigma^\dagger F^L_{\mu\nu} \Sigma F^{R\mu\nu} + H_1 \mathrm{Tr}(F^R_{\mu\nu} F^{R\mu\nu} + F^L_{\mu\nu} F^{L\mu\nu}) \ . \end{aligned} \tag{5.89}$$

All the parameters can be determined by experimental data. For instance, $L_1 = 0.4 \pm 0.3$, $L_2 = 1.35 \pm 0.3$, $L_3 = -3.5 \pm 1.1$, $L_4 = -0.3 \pm 0.5$, $L_5 = 1.4 \pm 0.5$, $L_6 = -0.2 \pm 0.3$, $L_7 = -0.4 \pm 0.2$, $L_8 = 0.9 \pm 0.3$, $L_9 = 6.9 \pm 0.7$, and $L_{10} = -5.5 \pm 0.7$. All have the unit of 10^{-3}.

5.6 Banks-Casher Formula and Vafa-Witten Theorem

How does the chiral symmetry breaking happen microscopically? For many years, physcists have relied on models to understand this. For instance, the Nambu-Jona-Lasinio model has been used to describe what drives the spontaneous symmetry breaking, much like the theories for superconductivity. However, in the context of QCD, our present understanding of the SSB comes from the Banks-Casher formula.

Before we discuss this formula, it is useful to have an introduction to formulation of QCD in Euclidean space. The Euclidean formulation has the advantage that many path integral expressions become real and positive definite. We introduce $x_4 = ix^0$, and $p_4 = ip_0$, and so the Minkowski invariant xp becomes $-\sum_{i=1}^4 (x_i p_i)$, a Euclidean invariant. We also introduce $A_4 = iA_0$ and the Lorentz condition becomes,

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 \ . \tag{5.90}$$

The electric field $\vec{E}$ in the Euclidean space is the imaginary of that in the Minkowski space and so $E^2 \to -E^2$, and $F^{\mu\nu}F_{\mu\nu} = -2(E^2 - B^2) \to 2(B^2 + E^2) = F_{\mu\nu}F_{\mu\nu}$. We also define the Euclidean version of the γ matrix with $\gamma_4^E = \gamma^0$ and $\gamma_i^E = -i\gamma^i$ and the commutators now become

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu} \tag{5.91}$$

The new γ matrices are hermitian. The QCD lagrangian is now

$$\mathcal{L}_{QCD} = -\left[\overline{\psi}(x)(\gamma_\mu D_\mu + M)\psi(x) + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}\right] \tag{5.92}$$

Notice that $\gamma_\mu D_\mu$ is now an antihermitian operator. We can define the Euclidean L to absorb the minus sign.

Consider now the exponential factor $\exp(iS)$ in the path integral. After rotation, the integral $\int d^4x$ becomes $-i\int d^4x$. The $-i$ here cancels the i in front of the iS and define the Eulidean action as

$$S_E = -\int d^4x\mathcal{L} \tag{5.93}$$

Therefore the integration meansure becomes $\exp(-S_E)$

Let us see how the spontaneous symmetry breaking takes place in QCD. To this goal, we need to introduce an explicit breaking of the symmetry. For example, we give a small mass to quarks. Consider the expectation value of $\langle\bar{u}u\rangle$. We write

$$\begin{aligned}\langle\overline{u}u\rangle &= -\int \frac{1}{V_4}d^4x\langle u(x)\bar{u}(x)\rangle \\ &= -\int[DA]e^{-S_{YM}}\text{Det}(\not{D} + M)\frac{1}{ZV_4}\text{Tr}[\frac{1}{\not{D} + m_u}] \ .\end{aligned} \tag{5.94}$$

where Tr is over spatial, color, and spin indices. Now consider the eigenstates of $\not{D}$. Because it is an anti-hermitian operator, we have

$$\not{D}|\lambda\rangle = i\lambda|\lambda\rangle \ , \tag{5.95}$$

where λ is real. The different $|\lambda\rangle$ are orthogonal and therefore we have

$$\text{Tr}[\frac{1}{\not{D} + m_u}] = \sum_i \frac{1}{i\lambda_i + m_u} \ . \tag{5.96}$$

On the other hand, we have $\text{Tr}(D + M) = \text{Tr}(-D + M)$ because $(\gamma^5)^2 = 1$. We get then

$$\text{Tr}\frac{1}{D + M} = \sum_i[\frac{1}{i\lambda_i + m_u} + \frac{1}{-i\lambda_i + m_u}] = \sum_i \frac{m_u}{m_u^2 + \lambda_i^2} \tag{5.97}$$

Intrduce now a $\delta(\lambda - \lambda_i)$ and integration over λ. We have then

$$\langle\bar{u}u\rangle = -\int d\lambda\rho(\lambda)\frac{m_u}{m_u^2 + \lambda^2} \tag{5.98}$$

where

$$\rho(\lambda) = \frac{1}{ZV_4}\int[DA]\exp(-S_{YM})\text{Det}(\not{D} + M)\sum_i 2\delta(\lambda - \lambda_i) \tag{5.99}$$

Before one takes the limit of $V \to \infty$, there is no $\lambda = 0$ and when $m_u \to 0$, the condestante vanishes. However, in the limit of $V \to \infty$, ρ develops a density at $\lambda = 0$, in fact, we find in the limit of $m_u \to 0^+$, we have

$$\langle \bar{u}u \rangle = -\pi\rho(0) \tag{5.100}$$

This is the Cashier-Banks formula which says when there is a spontaneous symmetry breaking, the quark energy-level has non-zero density at $\lambda = 0$. If $m_u \to 0^-$, then condestate has a positive sign. In any case, $m_q\langle 0|\bar{u}u|0\rangle$ is negative.

By studyiny the instantons, one finds the near zero eigenstates come from the so-called instantons, and their interactions.

It can be shown that in QCD with three light quark flavors, the chiral symmetry must be spontaneously broken. This can be done after the discussion of t' Hooft anomaly matching. For QCD with two light flavors, the answer is not so clear although the experimental evidence indicates that the spontaneous chiral symmetry breaking does happen. On the other hand, certain symmetries in theories like QCD cannot be spontaneously broken. In this section, we discussed the Vafa-Witten theorems which says that the vector symmetry in the massive QCD-like theory cannot be broken spontaneously. Moreover, the parity in QCD cannot be broken spontaneously eithers. Both results sometimes are called Vafa-Witten theorems.

Let us consider the expectation value of the quark fields

$$\begin{aligned}&\langle T\psi_{\alpha_N i_N}(x_N)\cdots\psi_{\alpha_1 i_1}(x_1)\bar{\psi}_{\beta_1 j_1}(y_1)\cdots\bar{\psi}_{\beta_N j_N}(y_N)\rangle \\ &= \frac{1}{Z}[DA]\mathrm{Det}(\not{D}+M)\exp(-S_{YM}) \\ &\left[(\not{D}+M)^{-1}_{x_1\alpha_1 i_1;y_1\beta_1 j_1}\cdots(\not{D}+M)^{-1}_{x_N\alpha_N i_N;y_N\beta_N j_N} + \mathrm{perm.}\right]\end{aligned} \tag{5.101}$$

where perm. means N! terms come from different contraction of the quark fields with their conjugators. The Green's function above has manifest $SU(N)$ symmetry if the quark masses are degenerate. However, this is not the point. The real point of proofing that there is no spontaneous breaking of the vector symmetry is to show when an external violation of the symmetry is introduced, it does not induces a sigular response from the system.

What one has to show is that the Green's function is bounded so that when a symmetric breaking happens. In fact, it can be shown that

$$G < N![\sum_i \frac{1}{|m_i|}^N \tag{5.102}$$

Therefore when the symmetry term δM is not zero, the induces change in G is still linear in δM. Therefore SSB never happens here.

Vafa and Witten has also show that the parity cannot be broken spontaneously in the vector-like gauge theory like QCD. What does it mean by SSB of parity? Although the lagrangian is invariant under parity but the vacuum state $|0\rangle = \alpha|+\rangle + \beta|-\rangle$ is not an eigenstate of parity. Then it is easy to show that $|0'\rangle = \alpha|+\rangle - \beta|-\rangle$ is also a possible vacuum state. Using orthogonality condition, we determine $|\alpha|^2 = 1/2$, $|\beta|^2 = 1/2$. So we have two degenerate vacua. An observable effect is that a parity-odd (equivalent of a non-singlet) operator $\hat{O}$ $(POP^{-1} = -\hat{O})$ now has a non-zero expectation value in the vacua: $\langle 0|\hat{O}|0\rangle \neq 0$ (vaccum condensate). The expectation values in $|0\rangle$ and $|0'\rangle$, however, differs by a minus sign. How does one selected the right vacuum in the path integral formulation? This is done by coaching the vacuum. [For quantities like ground state energy which are singlet operator, all vacuua yield the same answer.] Adding a term $\lambda\hat{O}$ to the hamiltonian. This term lift the degeneracy between the different vacuum. If $\lambda > 0$, the vacuum with negative $\langle O\rangle$ has lower energy. This state is selected as oppose to the other state. It is quite clear that $E(\lambda) < E(0)$.

Vafa and Witten's argument goes like this: Consider, for instance, a gluonic hermitian operator $F\tilde{F} = 1/2\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. In the Euclidean path integral, this operator always has an i because of the Wick rotation:

$$\exp -F(\lambda)V_4 = \int [DA]\mathrm{Det}(\not{D} + M)\exp\left(-S - i\int \lambda F\tilde{F}\right) \ . \tag{5.103}$$

When $\lambda = 0$, we have the usual free energy. When λ is non-zero, the exponential factor is always less than 1, $F(\lambda)$ is the alway larger at non-zero λ, i.e., the free energy reaches the minimum when $\lambda = 0$. This contradicts the SSB condition that $E(\lambda) < E(0)$.

However, the Vafa-Witten proof has a loophole.