Solutions to Final Take-home Exam

Physics 623, Spring 2010, O.W. Greenberg

(50 points) 1.1 (a) The degrees of freedom are space, spin and flavor. With all the quarks in the s-state the space wavefunction is totally symmetric, so the wavefunction must be antisymmetric under exchange of spin and flavor. Use Young diagrams and the hook rule. Combine the $SU(3)_{flavor}$ and $SU(2)_{spin}$ degrees of freedom into an $SU(6)_{flavor-spin}$ 6. Then we need $6 \otimes 6 \otimes 6$. Find this in steps. $6 \otimes 6 \to 21 \oplus 15$, with 21 symmetric and 15 antisymmetric.
$15 \otimes 6 \to 70 \oplus 20$, with 20 antisymmetric, so we want 20.
(b) $6 \to (3, 2)$ under $SU(6) \to SU(3)_{flaovr} \otimes (SU(2)_{spin}$. Then
$6 \otimes 6 \to (3, 2) \otimes (3, 2) \to [(6 + \bar{3}, 3 + 1)] \to (6, 3) \oplus (\bar{3}, 1) \oplus (6, 1) \oplus (\bar{3}, 3)$. The last two of these are antisymmetic, so we want them. Then $[(6, 1) \oplus (\bar{3}, 3)] \otimes (3, 2) \to$ $(10 + 8, 2) \ \oplus (8 + 1, 4 + 2) \to (10, 2) \oplus (8, 2) \oplus (8, 4) \oplus (1, 2) \oplus (8, 2) \oplus (1, 4)$. We want $(8, 2) \oplus (1, 4)$.
(c) The magnetic moment of a baryon B is

$$\mu_B = \langle B_\uparrow | \mu_3 | B_\uparrow \rangle,$$

where

$$\mu_3 = 2\mu_0 \sum_q Q_q S_q, \quad Q_q = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$$

(u, d, s) and S_q is the z-component of the spin of quark q. is the charge of the quarks.

For the magnetic moments of the proton and neutron, use the fact that the proton is uud and the neutron is udd. This problem can be solved using wave functions for the proton and neutron with the proper symmetry. The solution below uses creation operators which give a shorter way to do the calculation. Since we found in (b) that quarks are in the antisymmtric 20 of $SU(6)$ we can use fermi operators that anticommute to construct the proton and neutron. Then with $u^\uparrow$, $u^\downarrow$, $d^\uparrow$, $d^\downarrow$ anticommuting operators, the proton with spin up is

$$|p^\uparrow\rangle = |u^\uparrow u^\downarrow d^\uparrow\rangle$$

so the magnetic moment of the proton is carried by the d quark sinse the u quark contributions cancel since their spins are opposite. Then

$$\mu_p = 2\mu_0(-\frac{1}{3})(\frac{1}{2}) = -\frac{1}{3}\frac{e\hbar}{2mc}$$

where the first factor is the g factor, the second is the Bohr magneton of a particle of mass m and charge e, the third is the charge of the d quark in units of e and the last is the z component of the spin of the d quark.
The neutron with spin up is

$$|n^\uparrow\rangle = |u^\uparrow d^\downarrow d^\uparrow\rangle$$

so the magnetic moment of the neutron is carried by the u quark. Then

$$\mu_n = 2\mu_0(\frac{2}{3})(\frac{1}{2}) = \frac{2}{3}\frac{e\hbar}{2mc}$$

where, again, the first factor is the g factor, the second is the charge of the u quark in units of e the third is the charge of the u quark and the last is the z component of the spin of the u quark.
Note that the magnetic moments here point the wrong way!
1.2 (d) Similar arguments give 56.
(e) $56 \to (8,2) \oplus (10,4)$.
(f) With the $SU(3)_{color}$ 1_{color} antisymmetric wavefunction factored out, we can take the quarks as bosons carrying flavor and spin. The combination $u^\uparrow d^\downarrow - u^\downarrow d^\uparrow$ is an $I=0, S=0$ core so the proton with spin up is (I will omit †s for the creation operators)

$$|p^\uparrow\rangle = \frac{1}{\sqrt{3}} u^\uparrow(u^\uparrow d^\downarrow - u^\downarrow d^\uparrow)|0\rangle$$

and the neutron is

$$|n^\uparrow\rangle = \frac{1}{\sqrt{3}} d^\uparrow(u^\uparrow d^\downarrow - u^\downarrow d^\uparrow)|0\rangle.$$

(f) The magnetic moment of the proton is

$$\langle p^\uparrow \mu_3 | p^\uparrow\rangle = 2\mu_0\frac{1}{3}\{2[(\frac{2}{3}\frac{1}{2}+\frac{2}{3}\frac{1}{2}+(-\frac{1}{3})(-\frac{1}{2})]+[(\frac{2}{3})(\frac{1}{2})+(\frac{2}{3})(-\frac{1}{2})+(-\frac{1}{3})(\frac{1}{2})]\} = \frac{e\hbar}{2mc}$$

The magnetic moment of the neutron is

$$\langle n^\uparrow \mu_3 | n^\uparrow\rangle = 2\mu_0(\frac{1}{3})\{[(-\frac{1}{3})(\frac{1}{2})+(\frac{2}{3})(\frac{1}{2})+(-\frac{1}{3}(-\frac{1}{2})]+2[(-\frac{1}{3})(\frac{1}{2})+(\frac{2}{3})(-\frac{1}{2})+(-\frac{1}{3})\frac{1}{2}]\} = (-\frac{2}{3})\frac{e\hbar}{2mc}$$

The ratio $\mu_p/\mu_n = -3/2$ is accurate to about 3%, much better than we would expect.

(50 points) 2. (a) The elastic cross section in terms of partial waves is

$$\sigma_{elastic} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l.$$

At resonance $\delta_l = \pi/2$, so for a d-wave, $l = 2$, at resonance the maximum cross section is

$$\sigma_2 = \frac{20\pi}{k_{res}^2} = \frac{10\pi\hbar^2}{mE_{res}}$$

You don't have to replace k_{res} using $E_{res} = \hbar^2 k_{res}^2/2m$. Either answer will do.

(b) Sakurai gives $\cot \delta \approx (-2/\Gamma)(E - E_{res})$ near E_{res}. So

$$\delta \approx \frac{\pi}{2} + \frac{2}{\Gamma}(E - E_{res})$$

near E_{res}.

(c) The general expression for the elastic scattering amplitude in terms of partial waves is

$$f(\theta) = \sum_l (2l+1) f_l P_l(\cos\theta)$$

where there are various equivalent formulas for f_l. The most convenient one here is

$$f_{2\ res}(k) = \frac{1}{k \cot \delta_2 - ik} = \frac{1}{k(-\frac{2}{\Gamma}(E - E_{res}) - i)}$$

Then

$$f(\theta) = \frac{5}{k(-\frac{2}{\Gamma}(E - E_{res}) - i)} P_2(\theta) + f_{nonres}(\theta),$$

(d) The elastic cross section is

$$\sigma_{elastic} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta)|^2 d\Omega,$$

We use the orthogonality of the Legendre polynomials,

$$\int P_l(\cos\theta)P_{l'}(\cos\theta)d\Omega = \frac{2}{2l+1}\delta_{l,l'},$$

so that the cross terms vanish and

$$\sigma_{elastic} = \frac{5\pi\Gamma^2}{k^2((E-E_{res})^2+(\Gamma/2)^2} + 40\pi Re(f_{2\ res}f^{\star}_{2\ nonres}) + \sum_{l\neq 2}(2l+1)|f_{2\ nonres}|^2$$

(e) The optical theorem is $\sigma_{elastic} = (4\pi/k)Imf_{elastic}(0)$.

$$Imf(0) = \frac{5\Gamma^2}{4k((E-E_{res})^2+(\Gamma/2)^2)} + Imf_{nonres}(0).$$

We get agreement only for the first terms in (d) and (e), so the optical theorem checks only for the elastic part of the problem.
(f) This problem seems to imply $\sigma_{elastic} = \sigma_{total}$ for the resonant cross section, since introducing a real phase shift δ_2 tacitly implies that only elastic scattering occurs. Instead, to allow inelastic scattering, write that scattering from a spherically symmetric potential has a d-wave partial amplitude $f_2(k)$ with a pole at $E = E_{res} - i\Gamma/2$ and a complex residue $A(k)$. This form allows inelastic scattering since the condition of (g) below need not be satisfied, in contrast to the form given in (c) above which obeys (g).
(g) $|S_l(k)| = 1$, or $|1 + 2ikf_l(k)| = 1$, or $Imf_l(k) = k|f_l(k)|^2$.
(h) Levinson's theorem states $\delta_l(\infty) - \delta_l(0) = n_l$, where n_l is the no. of bound states for angular momentum l. Here $n_d = 4$.