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Physics 603 **HOMEWORK ASSIGNMENT #5** Spring 2014

Due date for problems: Tuesday, March 11 [deadline on March 13].
Reminder: midterm March 27.

1. (10) P&B problem 4.3. NB: the probability distribution in question is the binomial distribution.

2.(5) P&B problem 4.5. Recall that $q = \ln \mathcal{Z}$, where $\mathcal{Z}$ is the grand partition function.

3.(5) P&B problem 4.6 For the first part just take $G(T,p,N) = -k_B T \ln Y_N(p,T)$ and show that

$$(\partial G/\partial p)_{T,N} = \langle V \rangle$$

4.(10) Kardar problem 4.9 (cf. P&B 4.10 and 4.11). Kardar's $\mathcal{Q}$ is our $\mathcal{Z}$; $\mathcal{G}$ is Φ.

Langmuir isotherms: an ideal gas of particles is in contact with the surface of a catalyst.

(a) Show that the chemical potential of the gas particles is related to their temperature and pressure via $\mu = k_B T \left[\ln\left(P/T^{5/2}\right) + A_0\right]$, where A_0 is a constant.

(b) If there are $\mathcal{N}$ distinct adsorption sites on the surface, and each adsorbed particle gains an energy ϵ upon adsorption, calculate the grand partition function for the two-dimensional gas with a chemical potential μ.

(c) In equilibrium, the gas and surface particles are at the same temperature and chemical potential. Show that the fraction of occupied surface sites is then given by $f(T, P) = P/\left(P + P_0(T)\right)$. Find $P_0(T)$.

(d) In the grand canonical ensemble, the particle number N is a random variable. Calculate its characteristic function $\langle \exp(-ikN) \rangle$ in terms of $\mathcal{Q}(\beta\mu)$, and hence show that

$$\langle N^m \rangle_c = -(k_B T)^{m-1} \left.\frac{\partial^m \mathcal{G}}{\partial \mu^m}\right|_T,$$

where $\mathcal{G}$ is the grand potential.

(e) Using the characteristic function, show that

$$\left\langle N^2 \right\rangle_c = k_B T \left.\frac{\partial \langle N \rangle}{\partial \mu}\right|_T.$$

5.(10) Qualifier problem, next page, on statistical physics of money. Should be easy after Prof. Yakovenko's lecture!

5. UMD qualifier problem, January 2004: Probability distribution of money.

This problem explores an analogy between the Boltzmann-Gibbs probability distribution of energy in statistical physics and the probability distribution of money in a closed system of economic agents. Consider a system consisting of $N \gg 1$ economic agents. At a given moment of time, each agent i has a non-negative amount of money $m_i \geq 0$ (debt is not permitted). As the agents engage in economic activity, money is constantly transferred between the agents in the form of payments. However, the total amount of money is conserved in binary transactions between agents: $m_i + m_j = m_i' + m_j'$. This condition is analogous to conservation of energy in collisions between atoms in a gas. We also assume that the system is closed, so the total amount of money in the system M is also conserved.

We wish to obtain a formula for the money distribution function $P(m)$ for the system in statistical equilibrium. It is defined so that $P(m)\,dm$ is the fraction of agents with money in the interval $[m, m+dm]$ and satisfies the standard normalization conditions:

$$\int_0^\infty P(m)\,dm = 1, \qquad \int_0^\infty m\,P(m)\,dm = \langle m \rangle = M/N, \tag{1}$$

where $\langle m \rangle = M/N$ is the average amount of money per agent.

(a) Let us divide the money semi-axis $m \geq 0$ into equal intervals Δm and count the number of agents belonging to each interval: N_1, N_2, $N_3, \ldots$ Obviously $\sum_{r=1}^{\infty} N_r = N$. This configuration can be realized in many different ways by moving agents between the intervals while preserving the occupation numbers N_1, N_2, $N_3, \ldots$ Write a combinatorial formula for the number of ways W a given distribution N_1, N_2, $N_3, \ldots$ can be realized.

(b) Using the Stirling approximate formula $\ln n! \approx n \ln n - n$ for $n \gg 1$, obtain an expression for $S = \ln W$, the entropy of the distribution.

(c) Using the method of Lagrange multipliers, find the configuration N_r^* that maximizes entropy S under constraints that the total number of agents is conserved and the total amount of money is conserved:

$$\sum_{r=1}^{\infty} N_r = N, \qquad \sum_{r=1}^{\infty} m_r N_r = M. \tag{2}$$

(d) Obtain the money distribution function $P(m)$ for the fraction of agents belonging to a given interval Δm: $P(m_r)\,\Delta m = N_r^*/N$. Determine the values of the Lagrange multipliers from the normalization conditions (1).

(e) Compare the obtained result for $P(m)$ with the Boltzmann-Gibbs formula for the probability distribution of energy $P(E)$ in physics. What is the analog of temperature in the system of economic agents? Compare with $\langle m \rangle$.