Physics 402 Take Home Exam
Due At 9:00 AM, Monday March 13, 2006

This exam is open notes and open book. You may also use Mathematica or other symbolic manipulation programs. **If you use Mathematica or a similar program you must include the output to get credit.** Do not seek outside help. (I trust you.) If you cannot do a section of the exam do not panic. The exam is written in such a way as you can often do a later section of a problem while missing earlier parts. To aid you in this, I will often ask you to show that something is true rather than asking for the answer. **To get credit you must show how you obtained your answer from the basic physical and mathematical principles. You may use formulae that we derived in class or in the book as a starting point. Since you have considerable time on this exam, I fully expect your answers to be clear.**

If you have questions you may e-mail me (cohen@physics.umd.edu) or call me at the office (301) 405-6117 or at home (301) 654-7702 (**Before 10:00 p.m.**)

1) At time t=0 a hydrogen atom is in the state $|\psi(0)\rangle = \frac{1}{2}|\psi_{1,0,0}\rangle - \frac{1}{2}|\psi_{2,1,1}\rangle + \frac{i}{2}|\psi_{2,0,0}\rangle - \frac{i}{2}|\psi_{3,2,1}\rangle$ where the state $|\psi_{n,l,m}\rangle$ is an energy eigenstate characterized by the usual quantum numbers *n, l,* and *m* and (0) in the ket indicates the time. You may neglect spin in this problem.

   a) If the z component of the angular momentum, L_z of the state $|\psi(0)\rangle$ is measured what is the probability the measured value will be $+\hbar$? What is the probability that it will be 0? What is the probability that it will be $-\hbar$? Briefly explain your reasoning.

   b) If the energy of the state $|\psi(0)\rangle$ is measured what is the probability the measured value will be E_0 (where E_0 is the ground state energy) ? What is the probability that it will be $\frac{E_0}{2}$? What is the probability that it will be $\frac{E_0}{4}$? Briefly explain your reasoning.

   c) Find the expectation value of the energy, the z component of the angular momentum and the square of the angular momentum in this state. That is find $\langle\psi(0)|\hat{H}|\psi(0)\rangle$, $\langle\psi(0)|\hat{L}_z|\psi(0)\rangle$ and $\langle\psi(0)|\hat{L}^2|\psi(0)\rangle$.

   d) What is the expectation value of the x-component of the angular momentum, $\langle\psi(0)|\hat{L}_x|\psi(0)\rangle$?

   e) Find the state as a function of time, *i.e.* find $|\psi(t)\rangle$

2) Consider a system of two particles one of which is spin one and the other is spin 1/2. They interact via a rotational invariant (scalar) Hamiltonian. In this problem the spatial degrees of freedom are irrelevant--- you may assume that each particle is fixed in some spatial wave function. Since the Hamiltonian is a scalar it commutes with all three components of the total angular momentum which in this case is the total spin of the system.

a) Show that there are a total of 6 possible spin states for the two particle system. (Hint: this is as easy it sounds!)

b) On general grounds involving the nature of angular momentum , the eigenvalues of the Hamiltonian must have degeneracies. In particular there should be one set of two degenerate states and one set of four degenerate states. Explain why.

c) It turns out that the most general Hamiltonian for such a system can be written as $\hat{H} = a\hat{1} + b\frac{\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2}{\hbar^2}$ where *a, b* are constants with the dimension of energy and $\hat{1}$ is the identity operator $\hat{\vec{s}}_1$ is the spin operator for the spin one particle and $\hat{\vec{s}}_2$ is the spin operator for the spin ½ operator. Find the energies for the set of two degenerate levels and the set of four degenerate levels in terms of *a, b.* Hint: You may find it helpful to think about the total spin of the system.

3) The matrix elements of $\hat{x}\ \hat{y}\ \hat{z}$ between energy eigenstates of the hydrogen atom play an important role in the calculation of the rate at which photons are emitted (for reasons that are beyond the scope of this course). Here we will work in the simplified limit where spin can be neglected and the proton will be assumed to be so heavy that it sits without moving in the center of the atom. As usual the states are labeled by three quantum numbers *n,l m* . The associated wave functions in spherical coordinates are $\psi_{nlm}(r,\theta,\phi) = \frac{u_n}{r} Y_l^m(\theta,\phi)$. We will consider matrix elements between the states $|n\,l\,m\rangle$ and the ground state is $|100\rangle$.

a) As a first step derive expressions for *x* , *y*, and *z* in terms of *r* and spherical harmonics. That is derive $z = \left(\frac{4\pi}{3}\right)^{1/2} r\, Y_1^0(\theta,\phi)$ and analogous expressions for *x* and *y*. You may use the expression for spherical harmonics in the book.

b) Use part a) show that

$$\langle n\,l\,m|\hat{x}|100\rangle = \left(\frac{1}{6}\right)^{1/2}\left(\int_0^\infty dr\,(u_n(r))^* r\, u_1(r)\right)\left(\int d\Omega\, \left(Y_l^m(\theta,\phi)\right)^* \left(Y_1^{-1}(\theta,\phi) - Y_1^1(\theta,\phi)\right)\right)$$

$$\langle n\,l\,m|\hat{y}|100\rangle = i\left(\frac{1}{6}\right)^{1/2}\left(\int_0^\infty dr\,(u_n(r))^* r\, u_1(r)\right)\left(\int d\Omega\, \left(Y_l^m(\theta,\phi)\right)^* \left(Y_1^{-1}(\theta,\phi) + Y_1^1(\theta,\phi)\right)\right)$$

$$\langle n\,l\,m|\hat{z}|100\rangle = \left(\frac{1}{3}\right)^{1/2}\left(\int_0^\infty dr\,(u_n(r))^* r\, u_1(r)\right)\left(\int d\Omega\, \left(Y_l^m(\theta,\phi)\right)^* Y_1^0(\theta,\phi)\right)$$

where as usual Ω is the solid angle *i.e.* where $\int d\Omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin(\theta)$

c) Show that
$\langle n\,l\,m|\hat{x}|100\rangle = 0$ unless l=1 and m=1 or m= -1
$\langle n\,l\,m|\hat{y}|100\rangle = 0$ unless l=1 and m=1 or m= -1
$\langle n\,l\,m|\hat{z}|100\rangle = 0$ unless l=1 and m=0

These results are examples of *selection rules*. These give conditions which matrix elements must have to be nonzero. They play a central role in the absorption and emission of photons. The results of part c) imply for example that the dominant method by which photons are emitted (so-called electric dipole radiation) can only land an atom in the ground state if the state emitting the photon is in a p-wave state (*i.e.* an l=1 state).

4) A spin ½ particle interacts with an external potential. All of the energy levels are non-degenerate. The ground state energy is denoted E_g, the first excite state is $E_1 = 2E_g$, the second excited state is $E_2 = 4E_g$, the third excited state is $E_3 = 7E_g$, the third excited state is $E_4 = 12E_g$. The analogous states are denoted $|g\rangle$, $|1\rangle$, $|2\rangle$, $|3\rangle$ and the $|4\rangle$. Note that these states represent the full state (*i.e.* both the spin and space degrees of freedom)

a) From the information provided above one can deduce the Hamiltonian of the system does not commute with the total angular momentum. Explain why. What does this tell you about whether the Hamiltonian is rotationally invariant?

Now suppose that we have two of these spin ½ particles (they are identical) both of which are in this potential. Moreover these particles are non-interacting in the sense that the Hamiltonian has no interaction between them.

b) How many energy eigenstates are there in this two particle system with an energy less than $12E_g$? Explicitly construct these states and state the energy of each state. Use the notation where the state $|a,b\rangle$ indicates that the first particle is in state *a* and the second in state *b*, *eg.* $|g,1\rangle$ means that the first particle is in the ground state and the second is in the first excited state. Hint: Does the spin of the particles play any role?

5) In class we have studied the time evolution of a spin ½ particle in a magnetic field. We considered the case where the magnetic field was oriented along the z direction. This was convenient since we defined our basis states in terms of their *m* quantum number (which specifies the z-component of angular momentum). In this problem, I will ask you to do things *inconveniently.* Namely we will still work with our standard basis state $|\uparrow\rangle$ and $|\downarrow\rangle$ which correspond to spin up or down along the z axis, but this time the magnetic field is aligned in the x direction. Thus, the Hamiltonian is given by $\hat{H} = -2\Omega\,\hat{s}_x$ where

$\Omega = \frac{\mu_0 B}{\hbar}$. We may generally represent the state of the system as $|\psi(t)\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle$ where $|a(t)|^2 + |b(t)|^2 = 1$.

a) Show that the time evolution for the coefficients *a(t)* and *b(t)* is given by the following equation: -

$$-\Omega\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a(t)\\ b(t)\end{pmatrix} = i\frac{d}{dt}\begin{pmatrix}a(t)\\ b(t)\end{pmatrix}$$

b) Verify mathematically that the solution to the equation in a) is given by

$$\begin{pmatrix}a(t)\\ b(t)\end{pmatrix} = c_1 e^{i\Omega t}\begin{pmatrix}\sqrt{1/2}\\ \sqrt{1/2}\end{pmatrix} + c_2 e^{-i\Omega t}\begin{pmatrix}\sqrt{1/2}\\ -\sqrt{1/2}\end{pmatrix}$$

where c_1 and c_2 are constants fixed by the initial conditions.

c) Explain briefly the physical significance of the two terms in terms in part b) of eigenvectors and eigenvalues of operators relevant to the problem.

d) Suppose that a time t=0 the system is in the state $\begin{pmatrix}a(0)\\ b(0)\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$ Find expressions for the expectation values $\langle\hat{s}_x\rangle$, $\langle\hat{s}_y\rangle$, $\langle\hat{s}_z\rangle$ as functions of time.

e) If part d) is done correctly $\langle\hat{s}_x\rangle$ will be independent of time. Explain on general physical grounds involving the Hamiltonian operator and $\hat{s}_x$ why this should be true.