Problem Set ---Due March 3

1) A Stern-Gerlach apparatus oriented along the z axis picks out a spin up state of an electron. The electron passes through a second Stern-Gerlach apparatus this one oriented along an axis which makes an angle θ relative to the z axis in y-z plane. What is the probability that this second Stern-Gerlach apparatus will find the electron in a spin up state?

2) The purpose of this problem is convince you that rotations in quantum mechanics about an axis $\hat{n}$ and through an angle θ are described by a rotation operator $\hat{R}(\theta) = e^{i\theta\,\hat{\vec{J}}\cdot\hat{n}/\hbar}$ where $\hat{J}$ is the quantum mechanical operator for the angular momentum(there is a notation infelicity here in that the hat has two meanings). If this is correct then a rotation of the about the z axis will rotate the $\hat{x}$ into a linear combination of $\hat{x}$ and $\hat{y}$: $\hat{R}^{+}(\theta)\hat{x}R(\theta) \equiv e^{-i\theta\,\hat{J}_z/\hbar}\,\hat{x}\,e^{+i\theta\,\hat{J}_z/\hbar} = \hat{x}\cos(\theta) + \hat{y}\sin(\theta)$. Verify that this relation is true. Hints: i) acting on $\hat{x}$ or $\hat{y}$ the angular momentum or relevance is the $\hat{L}_z = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$; ii) the relation holds if it holds at and θ =0 and if the derivative of the right hand side matchs that of the left hand side for all angles.

3) Consider rotations acting on spin ½ states. In that case $\hat{R}(\theta) = e^{i\theta\,\hat{\vec{s}}\cdot\hat{n}/\hbar} = e^{i\theta\,\hat{\vec{\sigma}}\cdot\hat{n}/2}$.
   a) As a first step show that $\left(\hat{\vec{\sigma}}\cdot\hat{n}\right)^2 = \hat{1}$.
   b) Show that $e^{i\theta\,\hat{\vec{\sigma}}\cdot\hat{n}/2} = \cos(\theta/2) + i(\hat{\vec{\sigma}}\cdot\hat{n})\sin(\theta/2)$. (Hint: expand both sides as a Taylor series in the angle and us a)
   c) If this correctly gives the rotations then a rotation around the z axis through θ acting on $\hat{\sigma}_x$ should give a linear combination of $\hat{\sigma}_x$ and $\hat{\sigma}_y$:
   $\hat{R}^{+}(\theta)\hat{\sigma}_x R(\theta) \equiv e^{-i\theta\,\hat{\sigma}_z/2}\,\hat{\sigma}_x\,e^{i\theta\hat{\sigma}_z/2} = \hat{\sigma}_x\cos(\theta) + \hat{\sigma}_y\sin(\theta)$

   Hint: use part b) to do this.

4) In magnetic resonance experiments, the set up involves a strong constant magnetic field in the z direction and a rotating magnetic field in the x-y plane:
   $\vec{B} = B_0\hat{z} + B'\left(\hat{x}\cos(\omega\, t) - \hat{y}\sin(\omega\, t)\right)$.
   a) Show that the time-dependent Schrödinger equation for the spinor of spin ½ particle in such a field is given by
   $$\left(-\gamma\, B_0\vec{\sigma}_z - \gamma\, B'\left(\vec{\sigma}_x\cos(\omega\, t) - \vec{\sigma}_y\sin(\omega\, t)\right)\right)\chi = 2i\frac{d\chi}{dt}.$$
   b) It is common to work "in a rotating frame": let $\chi = e^{i\hat{\sigma}_z\omega\, t/2}\,\chi'$. Use the results of problem 3 c) to show that the time-dependent Schrödinger equation for the

spinor in the rotating frame is given by $\left(-\gamma B_{eff}\vec{\sigma}_z - \gamma B'\vec{\sigma}_x\right)\chi' = 2i\frac{d\chi'}{dt}$ with $B_{eff} = B_0 - \frac{\omega}{\gamma}$.

Note that in this rotating frame the magnetic field looks constant. When $B_{eff} >> B'$ the effective magnetic field is essentially along the z axis an a spin up state will to good approximation remain spin up. However when B_{eff} goes to zero the magnetic field in the rotating frame is in the x direction and cause the spin to precess about the x axis---effectively flipping the spin from up to down continuously and causing the time average of the expectation value of $\hat{s}_z$ to vanish.

c) Use the results of the precession calculation that we did in class (or you found in the book) to show that if the particle starts at t=0 in a spin up state, the time averaged expectation value of $\hat{s}_z$ is given by $\overline{\langle\hat{s}_z\rangle} = \frac{\hbar}{2}\frac{\left(B_0 - \frac{\omega}{\gamma}\right)^2}{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + B'^2}$. This quantifies the previous discussion. Away from the resonance the system is polarized in the z direction. Exactly on resonance the polarization in the z direction averages to zero.