PHYS 402 Homework---Due March 11

1. In this problem we consider two spin ½ particles interacting via the Hamiltonian $\hat{H} = a\,\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2$ where a is a constant. The computation of the energies using total spin is straightforward. Show that there is a nondegenerate state (“a singlet”) with energy $-\frac{3a\hbar}{4}$ and triply degenerate state (“a triplet”) with energy $\frac{a\hbar}{4}$.
2. In the next two problems we will work through the system discussed in problem 1 the dumb way: using the $\left|s_1 s_2 m_1 m_2\right\rangle$ basis. The states in this basis can be denoted $\left|\uparrow\uparrow\right\rangle$ $\left|\uparrow\downarrow\right\rangle$ $\left|\downarrow\uparrow\right\rangle$ $\left|\downarrow\downarrow\right\rangle$

   a. Show that

   $$\hat{s}_{1z}\hat{s}_{2z}\left|\uparrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left|\uparrow\uparrow\right\rangle \qquad \hat{s}_{1z}\hat{s}_{2z}\left|\uparrow\downarrow\right\rangle = -\frac{\hbar^2}{4}\left|\uparrow\downarrow\right\rangle \qquad \hat{s}_{1z}\hat{s}_{2z}\left|\downarrow\uparrow\right\rangle = -\frac{\hbar^2}{4}\left|\downarrow\uparrow\right\rangle \qquad \hat{s}_{1z}\hat{s}_{2z}\left|\downarrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left|\downarrow\downarrow\right\rangle$$

   b. Show that

   $$\hat{s}_{1x}\hat{s}_{2x}\left|\uparrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left|\downarrow\downarrow\right\rangle \qquad \hat{s}_{1x}\hat{s}_{2x}\left|\uparrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left|\downarrow\uparrow\right\rangle \qquad \hat{s}_{1x}\hat{s}_{2x}\left|\downarrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left|\uparrow\downarrow\right\rangle \qquad \hat{s}_{1x}\hat{s}_{2x}\left|\downarrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left|\uparrow\uparrow\right\rangle$$

   c. Show that

   $$\hat{s}_{1y}\hat{s}_{2y}\left|\uparrow\uparrow\right\rangle = -\frac{\hbar^2}{4}\left|\downarrow\downarrow\right\rangle \qquad \hat{s}_{1y}\hat{s}_{2y}\left|\uparrow\downarrow\right\rangle = \frac{\hbar^2}{4}\left|\downarrow\uparrow\right\rangle \qquad \hat{s}_{1y}\hat{s}_{2y}\left|\downarrow\uparrow\right\rangle = \frac{\hbar^2}{4}\left|\uparrow\downarrow\right\rangle \qquad \hat{s}_{1y}\hat{s}_{2y}\left|\downarrow\downarrow\right\rangle = -\frac{\hbar^2}{4}\left|\uparrow\uparrow\right\rangle$$

3. In this problem we construct and diagonalize the Hamiltonian matrix:
   a. Using the results of problem 2. show that Hamiltonian in problem one can be expresses as the following matrix in the basis $\left|\uparrow\uparrow\right\rangle$ $\left|\uparrow\downarrow\right\rangle$ $\left|\downarrow\uparrow\right\rangle$ $\left|\downarrow\downarrow\right\rangle$: $H = \frac{a\hbar^2}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
   b. Find the eignenvalues of the preceding Hamiltonian and verify that there there are three degenerate eigenvalue of $\frac{a\hbar}{4}$ and one of $-\frac{3a\hbar}{4}$ as seen in problem 1.

4. This problem a concerns a system of of two particles particle one with spin ½ and particle two with spin 1. Suppose that the Hamiltonian for the system $\hat{H} = a\,\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2$ where a is a constant. The first problem concerns some general properties of the system
   a. Show on general grounds that there are six states in the Hilbert space.
   b. What are the possible total spins?
   c. Explain why one expects only two distinct energies. One level which is four-fold degenerate and another which is two fold degenerate.
   d. Use properties of the total spin to show that the four fold degenerate states have an energy of $\frac{a\hbar}{2}$ while the doubly degenerate state has an energy of $-a\hbar$.
   e. Write the state $\left|\,|\,S = 3/2\ m = 1/2\right\rangle$ in the $\left|s_1 s_2 m_1 m_2\right\rangle$ basis.