Lecture 37 Highlights

The theory of superconductivity, formulated by Bardeen, Cooper and Schrieffer (BCS), is based on the concept of Cooper pairing of two electrons. The Cooper pair calculation for two electrons added to a filled Fermi sea at zero temperature proceeds as follows. We ask the question: will there be a bound state of two electrons added to the filled Fermi sea if there is a weak attractive interaction between them? In other words, will the 2 electrons added to states at the Fermi Energy (E_F) have an eigenenergy that is less than $2E_F$?

We seek a solution to the 2-identical electron (Fermion) Schrödinger equation:

$$\left\{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1,\vec{r}_2)\right\}\Psi(1,2) = E\Psi(1,2), \tag{1}$$

for the eigenenergy E. The 2-particle wavefunction for two identical Fermions must be overall anti-symmetric, or in other words: $\Psi(2,1) = -\Psi(1,2)$. We treat the two electrons as if they are "free" but confined to a box, with periodic boundary conditions. This gives rise to single-particle solutions of the form $\exp\left[\pm i\vec{k}\bullet\vec{r}\right]$, which are "running wave" solutions. If we add two particles to the filled Fermi sea (which means that all states of energy less than E_F are occupied, then the minimum center-of-mass energy would have the two electrons in states of equal and opposite momentum: $\Psi(1,2) \propto \exp\left[+i\vec{k}\bullet\vec{r}_1\right]\exp\left[-i\vec{k}\bullet\vec{r}_2\right]$, and $\left|\vec{k}\right| \geq k_F$, where k_F is the Fermi k-vector, defined as $E_F = \hbar^2 k_F^2/2m$.

To construct the full wavefunction there are just two possibilities:

$$\Psi_{SA}(1,2) = \cos\left(\vec{k}\bullet\vec{r}\right)\left|00\right\rangle$$

$$\Psi_{AS}(1,2) = \sin\left(\vec{k}\bullet\vec{r}\right)\begin{cases}\left|11\right\rangle\\ \left|10\right\rangle\\ \left|1-1\right\rangle\end{cases},$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative coordinate. If there is a small attractive interaction between the electrons, then the symmetric space wavefunction ($\Psi_{SA}(1,2)$) will be favored. This wavefunction puts the two electrons into a spin singlet state. The general wavefunction for the two electrons allows them to occupy any pair of states ($\vec{k},-\vec{k}$) outside the Fermi sphere:

$$\Psi(1,2) = \sum_{k>k_F} g_k \cos\left(\vec{k}\bullet\vec{r}\right)\left|00\right\rangle,$$

where g_k is a weighting function that depends only on the magnitude of $\vec{k}$ and not its direction.

Putting this *ansatz* into the Schrödinger equation (1) leads, after some manipulations (including sum busting) to:

$$\left(E - 2\varepsilon_k\right)g_k = \sum_{k'} g_{k'} V_{k,k'},$$

where $\varepsilon_k = \hbar^2 k^2 / 2m$ is the single-particle energy and $V_{k,k'} \equiv \int d^3 r \, V(r) \exp\left[i\left(\vec{k} - \vec{k}'\right) \bullet \vec{r} \right]$ is the Fourier transform of the pairing potential from real space into momentum space. We take the Cooper approximation for the attractive interaction between the electrons in momentum space. Fröhlich showed that two electrons can attract each other through a well-timed interaction with the vibrations of the positive ion crystal lattice. This attraction is retarded in time and is said to be mediated by the lattice. The approximate attractive potential is modeled as:

$$V_{k,k'} = \begin{cases} -V \text{ when } E_F \le \varepsilon \le E_F + \hbar\omega_c \\ \quad 0 \text{ when } \varepsilon > E_F + \hbar\omega_c \end{cases}$$

This potential favors pairing which involves scattering from states $(\vec{k}, -\vec{k})$ to states $(\vec{k}', -\vec{k}')$ both of which are within a thin skin of thickness $\hbar\omega_c$ of the Fermi energy. The energy scale $\hbar\omega_c$ is the characteristic energy of the lattice vibrations, and will depend on the material.

With this simplification of the potential one can solve for the eigenenergy of the two-electron problem;

$$E = 2E_F - 2\hbar\omega_c e^{-2/(N(E_F V)}, $$

where we have made the "weak coupling" approximation that $N(E_F)V << 1$, where $N(E_F)$ is the density of electronic states at the Fermi energy. This result shows that there is a bound state of the electrons for arbitrarily weak pairing interaction V. The full BCS theory of superconductivity builds the Cooper pairing into the wavefunction for all the electrons in the metal, not just these two "special case" electrons. That result earned them a Nobel prize in physics (the second for Bardeen), and their theory has had a profound influence on theoretical physics since then.