Physics 401 Homework 4---Due September 30

1. In class we asserted that in quantum mechanics physical properties with Hermitian operators. These were defined an operator $\hat{A}$ as Hermitian operator if $\int dx\,\psi^* (\hat{A}\psi)$ is real for all square integrable wave functions ψ. In this problem, you will use this definition to derive the following properties of Hermitian operators:

   a. If $\int dx\ \psi_1^* (\hat{A}\psi_2) = \int dx\ (\hat{A}\psi_1)^* \psi_2$ for all square integrable wave functions ψ_1, ψ_2 then $\hat{A}$ is Hermitian.

   b. If $\hat{A}$ is Hermitian, then $\int dx\ \psi_1^* (\hat{A}\psi_2) = \int dx\ (\hat{A}\psi_1)^* \psi_2$ for all square integrable wave functions ψ_1, ψ_2. Hint: consider $\psi = \psi_1 + \psi_2$ and use the definition.

   Note that since the property $\int dx\ \psi_1^* (\hat{A}\psi_2) = \int dx\ (\hat{A}\psi_1)^* \psi_2$ is both necessary and sufficient for $\hat{A}$ to be Hermitian (as shown in a. and b.) this property is an alternative definition for Hermiticity.

   c. If $\hat{A}$ is Hermitian then $\hat{A}^2$ is also. Hint: consider $\psi = \hat{A}\phi$ and start with $\int dx\,\psi^* \psi$. which we know to be real and use the property in a).

   d. If $\hat{A}$ and $\hat{B}$ are Hermitian , then $\int dx\ \psi_1^* (\hat{A}\hat{B}\psi_2) = \int dx\ (\hat{B}\hat{A}\psi_1)^* \psi_2$ Hint use the property in part b. twice.

   e. If $\hat{A}$ and $\hat{B}$ are Hermitian , then $\hat{C} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is Hermitian. Hint: use part d.

   f. If $\hat{A}$ and $\hat{B}$ are Hermitian , then $\hat{C} = i\left[\hat{A}, \hat{B}\right]$ is Hermitian. Hint: use part d.

2. Use the general form of the uncertainty relation to show that $\sigma_{x^2}\sigma_p \geq \hbar\left|\langle x\rangle\right|$ for any wavefunction. That is show that $\left(\langle x^4\rangle - \langle x^2\rangle^2\right)\left(\langle p^2\rangle - \langle p\rangle^2\right) \geq \left|\hbar\langle x\rangle\right|^2$.

3. Show that in quantum mechanics $\dfrac{d\langle \mathrm{O}\rangle}{dt} = \dfrac{i\left\langle\left[\hat{H}, \hat{\mathrm{O}}\right]\right\rangle}{\hbar}$ where $\hat{\mathrm{O}}$ is a Hermitian operator associated with a physical observable that contains no explicit time dependence and $\hat{H}$ is the Hamiltonian operator. Hint: Use the Schrödinger equation in the form $\hat{H}\psi = i\hbar\dfrac{d\psi}{dt}$.