Physics 401---Due November 28

1. Suppose we have two physical observables S and T which in quantum mechanics are associated with Hermitian operators $\hat{S}$ and $\hat{T}$. These operators do not necessarily commute. Further, suppose that the system is in an eignenstate of $\hat{S}$ with eigenvalue s_n : $\hat{S}|\psi\rangle = s_n|\psi\rangle$. Suppose one were to make two consecutive measurements of the system, say first measure T and then S. There is some probability that the measurement would yield t_m when measuring T and then s_n when measuring S. We can denote this probability $P_{t_m s_n}$. Alternatively one could first measure S and obtian s_n and then measure T and obtain t_m . We can denote this probalitiy $P_{s_n t_m}$. Show that $P_{t_m s_n} = P^2_{s_n t_m}$. Recall that after a measurement the system is an eigenstate of the measured operator; the probability that one is in this state is the absolute value squared of the inner product state of the system and the eigenstate in question.

2. Consider the infinites square-well from 0 to a that we studied earlier in the semester. The state $|n\rangle$ represents the nth energy eigenstate of the system.

   a. Show that the operator $\hat{x}$ is given by $\hat{x} = \sum_{j,k} | j\rangle\langle k| \frac{2}{a}\int_0^a dx \sin\left(\frac{j\pi x}{a}\right) x \sin\left(\frac{k\pi x}{a}\right)$

   b. Show that operator $\hat{x}^2$ is given by
   $$\hat{x}^2 = \sum_{j,k} | j\rangle\langle k| \frac{2}{a}\int_0^a dx \sin\left(\frac{j\pi x}{a}\right) x^2 \sin\left(\frac{k\pi x}{a}\right)$$

   c. Show that the matrix elements for $\hat{x}$ are given by
   $$\langle j|\hat{x}|k\rangle = \begin{cases} \frac{1}{2}a \quad \text{for} \quad k = j \\ \frac{4jk\left(-1+(-1)^{j+k}\right)}{(j^2-k^2)^2\pi^2}a \quad \text{for} \quad i \neq j \end{cases} .$$

   d. Show that the diagonal matrix elements for $\hat{x}^2$ are given by
   $$\langle j|\hat{x}^2|j\rangle = \left(\frac{1}{3} - \frac{1}{2j^2\pi^2}\right)a^2 .$$

3. This problem explores the relationship between the diagonal matrix elements of the square of an operator to the off-diagonal matrix elements

a. Explain on general grounds why $\langle j|\hat{O}^2|j\rangle = \sum_k |\langle j|\hat{O}|k\rangle|^2$ where $\hat{O}$ is an operator and the set of states denoted $|j\rangle$ form an orthonormal basis. (Hint: First show that $|\langle j|\hat{O}|k\rangle|^2 = \langle j|\hat{O}|k\rangle\langle k|\hat{O}|j\rangle$)

b. From part a we have for $\hat{O} = \hat{x}$: $\langle j|\hat{x}^2|j\rangle = \sum_k |\langle j|\hat{x}|k\rangle|^2$. Verify numerically that this is true for $\langle 1|\hat{x}^2|1\rangle$ using the eigenbasis of states of the infinite square well from problem 1. Note that the sum is over an infinite number of states but it converges rapidly. How many states do you need to get accuracy to 1 part in 1000? How many to get accuracy to 1 part in 10000?

Griffihs 3.23, 3.24