Physics 401 Take Home Exam
Due At 10:00 AM, Monday October 24, 2005

This exam is open notes and open book. You may also use Mathematica or other symbolic manipulation programs. **If you use Mathematica or a similar program you must include the output to get credit.** Do not seek outside help. (I trust you.) If you cannot do a section of the exam do not panic. The exam is written in such a way as you can often do later section of a problem while missing earlier parts. To aid you in this, I will often ask you to show that something is true rather than asking for the answer. **To get credit you must show how you obtained your answer from the basic physical and mathematical principles. You may use formulae we derived in class or in the book as a starting point. Since you have considerable time on this exam, I fully expect your answers to be clear.**

If you have questions you may e-mail me (cohen@physics.umd.edu) or call me at the office (301) 405-6117 or at home (301) 654-7702 (**Before 10:00 p.m.**) I will be at a conference at Jefferson Lab, Newport News Virginia from Wednesday evening until Saturday night. I will try to respond by e-mail while I am gone but I cannot guarantee that I will be able to respond until Sunday. Please do not call me at home until Sunday. If you cannot complete the exam in a timely manner due to an inability to contact me, I will grant a reasonable extension.

This is an exam and not a homework assignment. The work you turn in must be yours alone.

Table of possibly useful integrals:

In all of these expressions λ may be taken to be a real positive number

$$\int_0^\infty dx\, \exp(-\lambda x^2) = \sqrt{\frac{\pi}{4\lambda}} \qquad \int_{-\infty}^\infty dx\, \exp(-\lambda x^2) = 2\int_0^\infty dx\, \exp(-\lambda x^2)$$

$$\int_0^\infty dx\, x \exp(-\lambda x^2) = \frac{1}{2\lambda} \qquad \int_{-\infty}^\infty dx\, x \exp(-\lambda x^2) = 0$$

$$\int_0^\infty dx\, x^2 \exp(-\lambda x^2) = \frac{\sqrt{\pi}}{4\lambda^{3/2}} \qquad \int_{-\infty}^\infty dx\, x^2 \exp(-\lambda x^2) = 2\int_0^\infty dx\, x^2 \exp(-\lambda x^2)$$

$$\int_0^\infty dx\, x^3 \exp(-\lambda x^2) = \frac{1}{2\lambda^2} \qquad \int_{-\infty}^\infty dx\, x^3 \exp(-\lambda x^2) = 0$$

$$\int_0^\infty dx\, x^4 \exp(-\lambda x^2) = \frac{3\sqrt{\pi}}{8\lambda^{5/2}} \qquad \int_{-\infty}^\infty dx\, x^4 \exp(-\lambda x^2) = 2\int_0^\infty dx\, x^4 \exp(-\lambda x^2)$$

$$\int_0^a dx\, \sin^4\left(\tfrac{\pi x}{a}\right) = \frac{3a}{8} \qquad \int_0^a dx\, \sin^2\left(\tfrac{\pi x}{a}\right)\sin\left(\tfrac{n\pi x}{a}\right) = \begin{cases} \dfrac{-4a}{(-4n+n^3)\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

1) **(20 pts.)** A beam of electromagnetic radiation with frequency ν is directed towards a hydrogen atom in its ground state. The radiation ionizes the hydrogen---that is, it knocks out the electron leaving behind a proton. You may assume that is a single photon process: one photon is absorbed by the hydrogen atom and its energy is used to ionize the atom. Find an expression for the magnitude of the momentum of the outgoing electron. Express your answer in terms of the frequency of the radiation ν, the mass of the electron, m_e, the magnitude of charge of the electron, *e,* and universal constants such as Planck's constant. You may assume Bohr's expression is correct for the energies of the hydrogen atom and that the proton mass is so large that it remains essentially stationary in the process. You may also assume that the final momentum of the electron is low enough so that a nonrelativistic expression for the energy is valid. *Hint: Energy conservation is central in this problem but momentum conservation is not; the proton is so heavy that it can recoil with a very small velocity taking away significant momentum but negligible energy.*

2) **(30 pts.)** Consider a quantum mechanical system describing a particle whose wave function is given by:

$\psi(x) = \left(\dfrac{32}{25\pi\, x_0^2}\right)^{1/4} \left(1+\frac{x}{x_0}\right)\exp\left(-\frac{x^2}{x_0^2}\right)$, where $x_0 > 0$; the wave function is properly normalized.

a) Show that for this system : $\langle x\rangle = \dfrac{2x_0}{5}$, $\langle x^2\rangle = \dfrac{7x_0^2}{20}$, $\langle p\rangle = 0$, $\langle p^2\rangle = \dfrac{7\hbar^2}{5x_0^2}$

b) Find the uncertainties in *x* and *p,* that is find σ_x and σ_p. Verify explicitly that the uncertainty principle for *x* and *p* is respected. *In this problem you may use the results of a) even if you did not get them.*

c) What is the probability that the particle will be found at positive x?

Hint: It is **much** easier to do this with a symbolic manipulation program such as Mathematica then by hand. Note that relevant integrals have no closed analytic form as indefinite integrals but can be directly evaluated as definite integrals. If you wish to do them by hand, a table of some useful integrals has been provided at the top of the exam.

3) **(50 pts.)** In this problem, consider the one-dimensional quantum mechanical harmonic oscillator; the Hamiltonian is $\hat{H} = \dfrac{1}{2m}\hat{p}^2 + \dfrac{1}{2}m\omega^2\hat{x}^2$. Suppose at time t=0 the wave function is given by

$\psi(x,t=0) = \dfrac{1}{\sqrt{10}}\psi_0(x) + \dfrac{3}{\sqrt{10}}\psi_2(x)$, where $\psi_0(x)$ and $\psi_2(x)$ are the ground and second excited states.

a) Show the wave function is correctly normalized (*i.e.* show that the total probability is one). *Hint: You do not need to explicitly evaluate any integrals to do this.*

b) Find the expectation value of the energy, $\langle \hat{H}\rangle$, for this state.). *Hint: You do not need to explicitly evaluate any integrals to do this.*

c) Show that $\psi(x,t) = e^{-i\omega t/2}\left(\dfrac{1}{\sqrt{10}}\psi_0(x) + \dfrac{3}{\sqrt{10}}\psi_2(x)e^{-i2\omega t}\right)$

d) In class we showed for harmonic oscillators that

$$\left\langle \psi_j \middle| \hat{x}^2 \, \psi_j \right\rangle = \int dx \, (\psi_j(x))^* \, \hat{x}^2 \, \psi_j(x) \; = \; \frac{(2j+1)\hbar}{2m\omega}.$$

Here I want you to show that

$$\left\langle \psi_{j+2} \middle| \hat{x}^2 \psi_j \right\rangle = \int dx \, (\psi_{j+2}(x))^* \hat{x}^2 \, \psi_j(x) \; =$$

$$\left\langle \psi_j \middle| \hat{x}^2 \psi_{j+2} \right\rangle = \int dx \, (\psi_j(x))^* \hat{x}^2 \, \psi_{j+2}(x) \; = \frac{\sqrt{(j+1)(j+2)} \;\; \hbar}{2m\omega}$$

Hint: use the ladder operators.

e) For the wave function in part c. specified above, find $\left\langle \hat{x}^2 \right\rangle$ as a function of time. *Hint:The results in parts c) and d) are very useful. They made be used regardless of whether you solved pars c) and d).*

4) (**40 pts.)** In this problem we will consider a particle which moves in one dimension and is confined to a one-dimensional box (an infinite square well) of length; for convenience we have located the box between *x=0* and *x=a* (*i.e.* the potential is given by $V(x) = \begin{cases} 0 & \text{for } 0 < \text{x} < a \\ \infty & \text{otherwise} \end{cases}$). Suppose at t=0 that the system starts in the state: $\psi(x,t=0) = \begin{cases} \sqrt{\frac{8}{3a}} \sin^2\left(\frac{\pi x}{a}\right) & \text{for } 0 < \text{x} < a \\ 0 & \text{otherwise} \end{cases}$

a) Verify that the wave function is correctly normalized (*i.e.* show that the total probability is one)

b) Show that the wave function at all times can be written as

$$\psi(x,t) = \sum_{n=1}^{\infty} \frac{-16}{\sqrt{3}\,\pi\left(-4(2n-1)+(2n-1)^3\right)} \psi_{(2n-1)}(x) \exp\left(-i \frac{\hbar(2n-1)^2\pi^2}{2ma^2} t\right), \text{ where } \psi_n(x) = \sqrt{\tfrac{2}{a}} \sin\left(\tfrac{n\pi x}{a}\right).$$

Hint: The integrals needed to find the coefficients in this expression may found using a symbolic manipulator or with the help of the integral table above.

c) Suppose that the energy is measured. Various possible energies may be measured and they will come up with calculable probabilities. Find the probability that the particle has energy:

i) $\dfrac{\hbar^2\pi^2}{2ma^2}$

ii) $\dfrac{4\hbar^2\pi^2}{2ma^2}$

iii) $\dfrac{9\hbar^2\pi^2}{2ma^2}$

iv) $\dfrac{11\hbar^2\pi^2}{2ma^2}$

v) Any energy not considered in i)-iv).

In problem c) numerically evaluate any formal expressions to get a sense of how large these probabilities are.

Hint: Some of these probabilities are zero; one of these probabilities is always zero for any wave function. Some of these probabilities are nonzero but very small.

5) **(20 pts.)** Consider a Hamiltonian operator $\hat{H}$ which represents the energy operator and another operator $\hat{\Theta}$ which corresponds to some generic observable Θ. We will assume in this problem that $\hat{\Theta}$ and $\hat{H}$ have no explicit time-dependence and that $\hat{H}$ has a discrete spectrum. The purpose of this problem is to find an expression for the time-averaged expectation of Θ. That is we want to find

$$\overline{\langle\Theta\rangle} \equiv \lim_{T\to\infty} \frac{\int_{-T}^{T} dt \langle\Theta\rangle}{2T} = \lim_{T\to\infty} \frac{\int_{-T}^{T} dt \left\langle \psi(t) \middle| \hat{\Theta}\psi(t) \right\rangle}{2T}$$

(where the line over the expectation value is a shorthand for time average. This problem is of importance because often in an experimental situation one must measure for long times to get good statistics for practical reasons associated with the speed of the detector. In these cases when one measures statistical averages they are time averaged quantities.

a) As a first step show that as a function of time $\langle\Theta\rangle = \left\langle \psi(t) \middle| \hat{\Theta}\psi(t) \right\rangle = \sum_{j,k} c_j^* c_k \exp\left(\frac{i(E_j - E_k)t}{\hbar}\right) \left\langle \psi_j \middle| \hat{\Theta}\psi_k \right\rangle$ where E_n and $\psi_n(x)$ represent the eigenvalues and eigenfunctions of $\hat{H}$ and $c_n = \langle \psi_n | \psi(t=0) \rangle = \int dx\, \psi_n^*(x)\, \psi(x, t=0)$.

b) From the definition of $\overline{\langle\Theta\rangle}$ and the result in a) show that $\overline{\langle\Theta\rangle} = \sum_n |c_n|^2 \left\langle \psi_n \middle| \hat{\Theta}\psi_n \right\rangle$. Discuss briefly the physical interpretation of this result in terms of probabilities of various energy measurements. Hint: recall *j, k* and *n* are dummy variables in the various expressions.