QUANTUM PHYSICS I

PROBLEM SET 3

due October 9

A. (Griffths 2.22) The gaussian wave packet

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2}, \tag{1}$$

where A and a are constants (and a is real and positive).

a) Normalize $\Psi(x,0)$.

b) Find $\Psi(x,t)$ Griffths gives away the answer so I might as well do the same:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+(2i\hbar at/m))}}{\sqrt{1+(2i\hbar at/m}}. \tag{2}$$

c) Find $|\Psi(x,t)|^2$ and express your answer in terms of

$$w = \sqrt{\frac{a}{1+(2\hbar at/m)^2}}. \tag{3}$$

Sketch $|\Psi|^2$ as a function of x at $t = 0$ andfor some large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on ?

d) Find $\langle x\rangle, \langle p\rangle, \langle x^2\rangle, \langle p^2\rangle, \sigma_x$ and σ_p. Partial answer: $\langle p^2\rangle = a\hbar^2$.

e) Does the uncertainty principle hold ? At what time does the the system come closest to the uncertainty limit ? Do you have a physical understanding of why the uncertainty evolves in time the way it does ?

B. All you wanted to know about Dirac's δ-function and were afraid to ask

We can define the δ-function buy its behavior inside integrals:

$$\int_a^b f(x)\delta(x-y) \equiv f(y), \quad \text{for} \quad a < y < b, \tag{4}$$

for any well-behaved function $f(x)$, usually called the *test-function*. Show that

a) $\int_{-\infty}^{\infty} dx(x^3 - 1)\delta(x-1) = 0$

b) $\delta(cx) = \frac{1}{|c|}\delta(c)$ (Hint: Insert both sides of the equation in the definition of δ above and change variables.)

c) $\frac{d\theta(x)}{dx} = \delta(x)$ where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x < 0. \end{cases} \tag{5}$$

(and $\theta(0) = 1/2$ if it ever matters).

d) What is the Fourier transform of $\delta(x)$

$$F(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx}\delta(x) = ? \tag{6}$$

Use Plancherel's theorem (see text) to show that

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}, \tag{7}$$

which is a relation we used in class after a hand waving "proof".

e) Another way of defining the δ-function is through the relation

$$\delta(x) \equiv \lim_{\alpha\to\infty} \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}. \tag{8}$$

Show that the result of d) is the same using this new definition. Feel free to exchange the order of limits and integrations and assume that the test functions are as well behaved as necessary.

C. Raising-and-lowering-operatorology, Griffths 2.12

Find $\langle x\rangle, \langle p\rangle, \langle x^2\rangle$ and $\langle p^2\rangle$ for the nth excited state of the harmonic oscillator. Is the uncertainty principle satisfied ? Hint: use a_+, a_-.