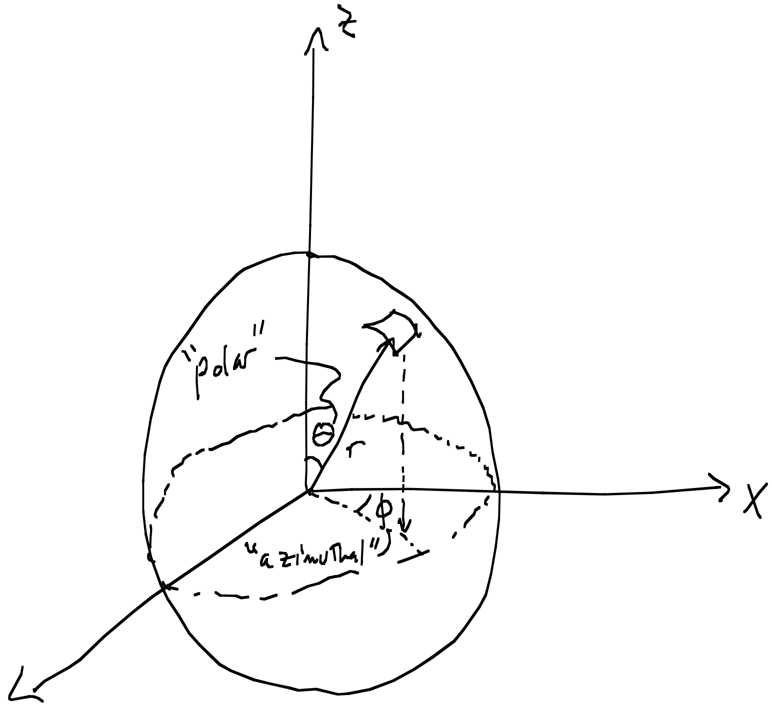
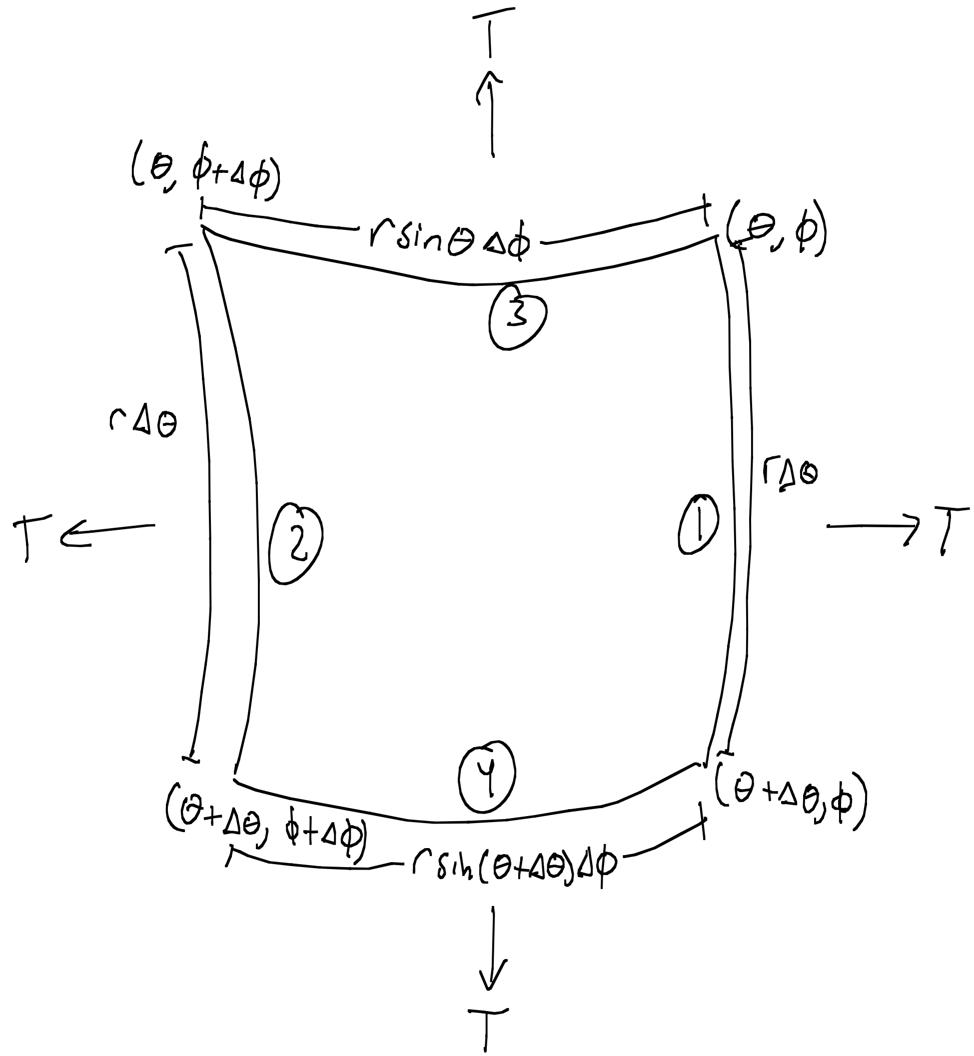
Classical waves on surface of a sphere

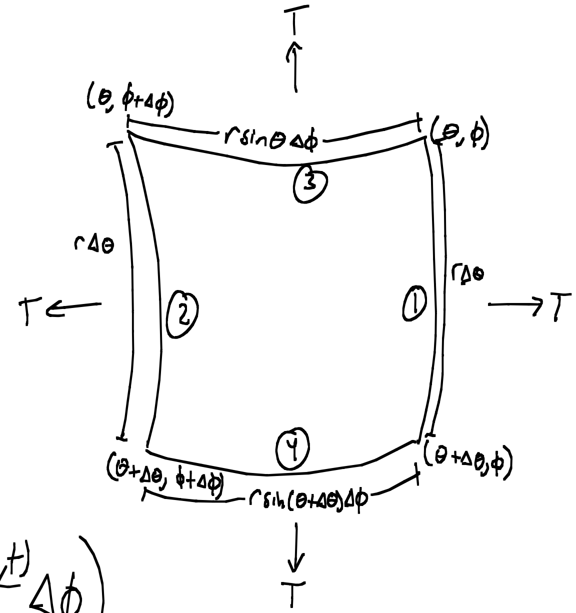




Newton's 2nd Law

$F = ma$

$$\textcircled{2} - \textcircled{1} + \textcircled{4} - \textcircled{3} = \rho\,(r\Delta\theta \cdot r\sin\theta\Delta\phi)\,\frac{\partial^2 f(\theta,\phi,t)}{\partial t^2}$$



$\textcircled{1}$: $$T r\Delta\theta \frac{\partial f(\theta,\phi,t)}{r\sin\theta\,\partial\phi}$$

$\textcircled{2}$: $$T r\Delta\theta \frac{\partial f(\theta,\phi+\Delta\phi,t)}{r\sin\theta\,\partial\phi} \sim \frac{T\Delta\theta}{\sin\theta}\frac{\partial}{\partial\phi}\left(f(\theta,\phi,t) + \frac{\partial f(\theta,\phi,t)}{\partial\phi}\Delta\phi\right)$$

$\textcircled{3}$: $$T r\sin\theta\Delta\phi \frac{\partial f(\theta,\phi,t)}{r\partial\theta}$$

$\textcircled{4}$: $$T r\sin(\theta+\Delta\theta)\Delta\phi \frac{\partial f(\theta+\Delta\theta,\phi,t)}{r\partial\theta} \sim T\Delta\phi\left(\sin\theta + \Delta\theta\frac{d\sin\theta}{d\theta}\right)\left(\frac{\partial}{\partial\theta}\left(f(\theta,\phi,t) + \Delta\theta\frac{\partial f(\theta,\phi,t)}{\partial\theta}\right)\right)$$

$$\sim T\Delta\phi\left(\sin\theta\frac{\partial f}{\partial\theta} + \Delta\theta\frac{\partial\sin\theta}{\partial\theta}\frac{\partial f}{\partial\theta} + \Delta\theta\sin\theta\frac{\partial^2 f}{\partial\theta^2}\right)$$

$$= T\Delta\phi\left(\sin\theta\frac{\partial f}{\partial\theta} + \Delta\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\right)$$

Equation of Motion

$$② - ① + ④ - ③ = \rho \left(r\Delta\theta \cdot r\sin\theta\Delta\phi \right) \frac{\partial^2 f(\theta,\phi,t)}{\partial t^2}$$

$$\frac{T}{\sin\theta} \frac{\partial^2 f}{\partial \phi^2} \cancel{\Delta\theta} \cancel{\Delta\phi} + T \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial \theta} \right) \cancel{\Delta\theta} \cancel{\Delta\phi} = \rho \left(\cancel{r\Delta\theta} \cdot r\sin\theta \cancel{\Delta\phi} \right) \frac{\partial^2 f(\theta,\phi,t)}{\partial t^2}$$

$$\frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta} \right) = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} \qquad \left(c^2 \equiv \frac{T}{\rho} \right)$$

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

This is a wave equation where $\nabla^2 \equiv \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial}{\partial\theta} \right]$