A more direct approach to Harmonic Oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

divide by $-\frac{\hbar\omega}{2}$

$$\frac{\hbar}{m\omega}\frac{d^2\psi}{dx^2} - \frac{m\omega}{\hbar}x^2\psi = -\frac{2E}{\hbar\omega}\psi$$

define $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ (unitless) such that $x = \sqrt{\frac{\hbar}{m\omega}}\,\xi$ and $dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi$:

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$

$\left(K \equiv \frac{2E}{\hbar\omega}\right)$. So far, we have only recast our Schrodinger eqn in a more managable, unitless form. We haven't gotten any closer to solving it Tho!

Asymptotic behavior

For large $\xi \gg 1$ (large x)

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2 \psi$$

This has approximate sol'n $\psi(\xi) = Ae^{-\xi^2/2} + \underbrace{\cancel{B}e^{+\xi^2/2}}_{=0 \text{ so that } \psi \text{ is normalizable.}}$

Check it: $\frac{d\psi}{d\xi} = -\xi A e^{-\xi^2/2}$, $\frac{d^2\psi}{d\xi^2} = A\left(\xi^2 e^{-\xi^2/2} - e^{-\xi^2/2}\right)$

$$A\left(\xi^2 e^{-\xi^2/2} - \underbrace{\cancel{e^{-\xi^2/2}}}_{\text{negligible for } \xi \gg 1}\right) \approx \xi^2 A e^{-\xi^2/2}$$

This suggests that all our solutions will be proportional to $e^{-\xi^2/2}$, since it will dominate for large ξ.

Ansatz: $\psi(\xi) = h(\xi)e^{-\xi^2/2}$

$$\frac{d\psi}{d\xi} = h' e^{-\xi^2/2} - \xi e^{-\xi^2/2} h$$

$$\frac{d^2\psi}{d\xi^2} = h'' e^{-\xi^2/2} - \xi e^{-\xi^2/2} h' - \left(h' \xi e^{-\xi^2/2} + h\left(e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2}\right)\right)$$

Schrodinger eqn becomes:

$$h'' e^{-\xi^2/2} - \xi e^{-\xi^2/2} h' - \left(h' \xi e^{-\xi^2/2} + h\left(e^{-\xi^2/2} - \cancel{\xi^2 e^{-\xi^2/2}}\right)\right) = (\cancel{\xi^2} - K)\, h e^{-\xi^2/2}$$

After eliminating common factor of $e^{-\xi^2/2}$,

$$\frac{d^2h}{d\xi^2} - 2\xi \frac{dh}{d\xi} + (K-1)h = 0$$

"Brute force"

Substitute: $h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$ so $\frac{dh}{d\xi} = \sum_{j=0}^{\infty} j a_j \xi^{j-1}$

and $\frac{d^2h}{d\xi^2} = \sum_{j=0}^{\infty} j(j-1) a_j \xi^{j-2} = \sum_{j=2}^{\infty} j(j-1) a_j \xi^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} \xi^j$

Then our eqn for $h(\xi)$ becomes:

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} \xi^j - \sum_{j=0}^{\infty} 2\xi j a_j \xi^{j-1} + \sum_{j=0}^{\infty} (K-1) a_j \xi^j = 0$$

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} - 2 j a_j + (K-1) a_j \right] \xi^j = 0$$

This can only be true if <u>all</u> coefficients of all terms are zero!

Recursion Relation

The coefficients can be written $(j+2)(j+1)a_{j+2} - (2j + 1 - K)a_j = 0$

This leads to: $$a_{j+2} = \frac{2j+1-K}{(j+2)(j+1)} a_j$$ "recursion relation"

For large $j \gg 1$, $a_{j+2} \approx \frac{2}{j} a_j \implies a_j \approx \frac{C}{(j/2)!}$

This gives $h(\xi) \approx \sum_{j=0}^{\infty} \frac{C}{(j/2)!} \xi^j \gtrsim \sum_{j=0}^{\infty} C \frac{(\xi^2)^j}{j!} = Ce^{+\xi^2}$ But, This is not normalizable!

So series must terminate for some $j_{max} = n$ so That $a_{j_{max}+2} = 0$

i.e. $h(\xi)$ is a polynomial ("Hermite polynomial")

$$2n + 1 - K = 0 \implies 2n + 1 - \frac{2E}{\hbar\omega} = 0$$

$$E = \hbar\omega\left(n + \tfrac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

Same as ladder operator method! ✓

Wave functions

For an eigenfunction, $K = 2n+1$. Therefore,

$$a_{j+2} = \frac{2j+1-(2n+1)}{(j+1)(j+2)} a_j = \frac{-2(n-j)}{(j+1)(j+2)} a_j , \quad j \leq n$$

We want to generate $h_n(\xi) = \sum_j^{\infty} a_j \xi^j$, and wavefunctions $\psi_n(\xi) = h_n(\xi) e^{-\xi^2/2}$

for $n=0$, $h_0(\xi) = a_0$, $\psi_0(\xi) \propto e^{-\xi^2/2}$

$n=1$, $h_1(\xi) = a_1 \xi$, $\psi_1(\xi) \propto \xi e^{-\xi^2/2}$

$n=2$, for $j=0$, $a_2 = -2a_0$ so $h_2(\xi) = a_0 - 2a_0\xi^2$, $\psi_2(\xi) \propto (1-2\xi^2)e^{-\xi^2/2}$

etc....