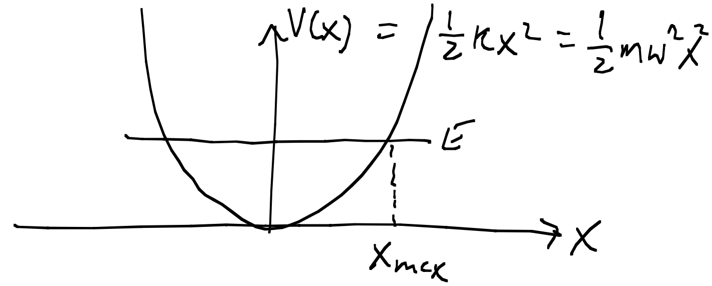
Classical Behavior of harmonic oscillator



$$F = -kx \longrightarrow V(x) = -\int F(x)\,dx = \frac{1}{2}kx^2$$

Newton's 2nd Law: $F = ma = m\ddot{x}$

$$\ddot{x}(t) = -\frac{k}{m}x(t) \qquad \text{soln: } x(t) = A\sin\omega t + B\cos\omega t, \quad \omega = \sqrt{\frac{k}{m}} \to k = m\omega^2$$

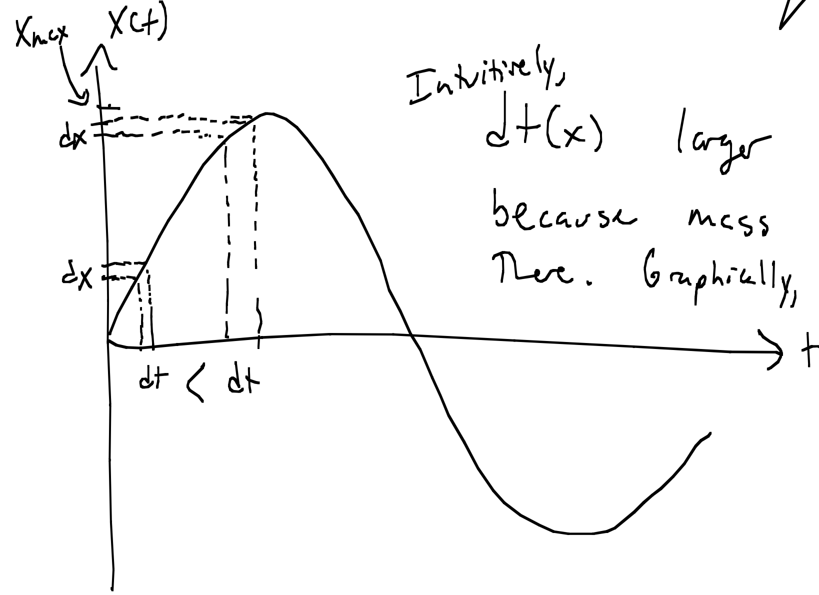
Let's arbitrarily choose initial conditions s.t. $B=0$. Then $A = x_{max}$,

The "classical turning point" where $E = V(x) = \frac{m\omega^2}{2}x_{max}^2 \longrightarrow x_{max} = \sqrt{\frac{2E}{m\omega^2}}$

Classical probability density can be obtained from this eqn. of motion by calculating $dt(x)$:

$$x(t) = x_{max}\sin\omega t$$

$$t(x) = \frac{\operatorname{asin}\frac{x}{x_{max}}}{\omega}$$



Intuitively, $dt(x)$ larger near x_{max}, because mass is moving slower there. Graphically, slope is lower.

Classical probability distribution

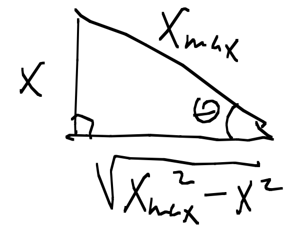
$$dt(x) = \frac{dt}{dx}\,dx = \frac{d}{dx}\left(\frac{\operatorname{asin}\frac{x}{x_{max}}}{\omega}\right)\cdot dx = \frac{dx}{\omega\, x_{max}\sqrt{1-\left(\frac{x}{x_{max}}\right)^2}} = \frac{dx}{\omega\sqrt{x_{max}^2-x^2}}$$

OR

$$= \frac{1}{\frac{dx}{dt}}\,dx = \frac{dx}{x_{max}\,\omega\cos\omega t} = \frac{dx}{x_{max}\,\omega\cos\left(\operatorname{asin}\frac{x}{x_{max}}\right)}$$

$$= \frac{dx}{\cancel{x_{max}}\,\omega\,\frac{\sqrt{x_{max}^2-x^2}}{\cancel{x_{max}}}} = \frac{dx}{\omega\sqrt{x_{max}^2-x^2}}$$

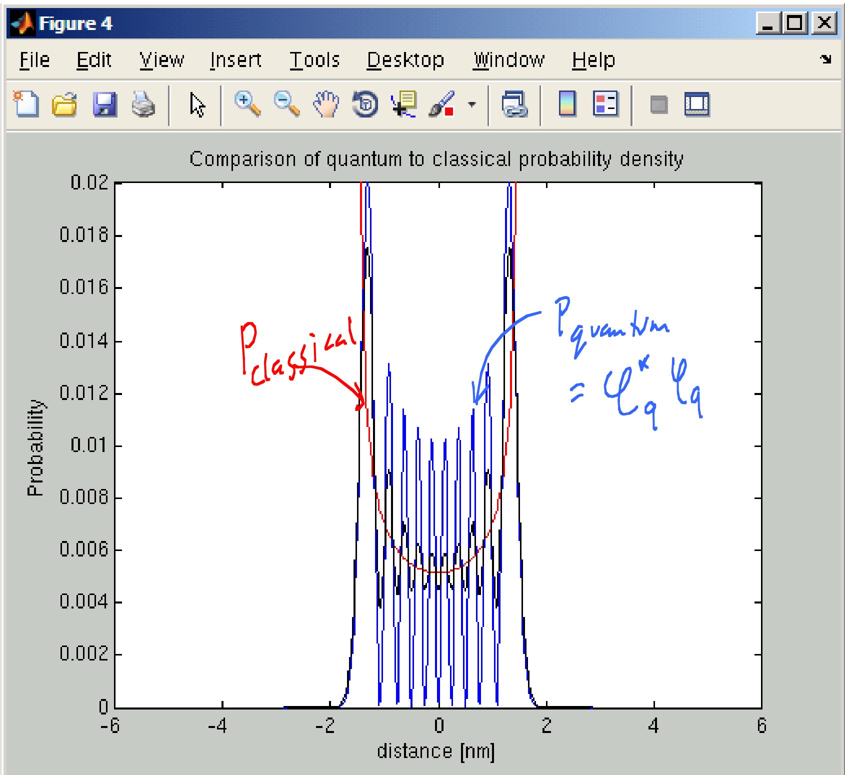
Same!

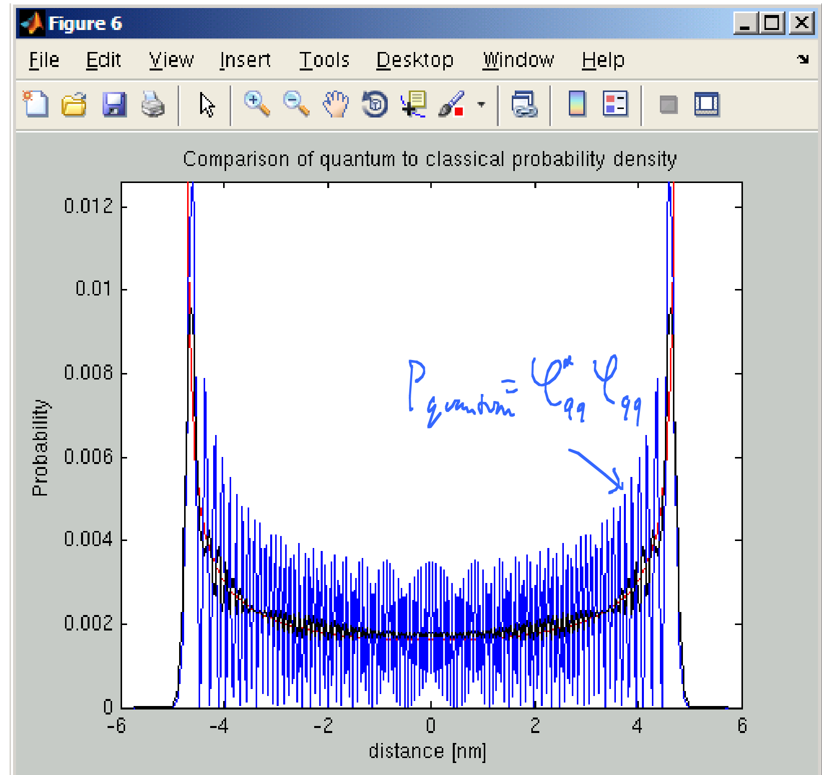


Probability density is $dt(x)$ normalized by half a period per dx

$$P_{classical} = \frac{1}{\pi\sqrt{x_{max}^2-x^2}} \Longleftrightarrow P_{quantum} = \varphi^*\varphi$$

Numerical Comparison





We can see the "correspondence principle" working here:

Quantum systems behave like their classical counterparts in the limit $n \to \infty$!

Harmonic Oscillator soln by "brute force"

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

divide by $-\frac{\hbar\omega}{2}$

$$\frac{\hbar}{m\omega}\frac{d^2\psi}{dx^2} - \frac{m\omega}{\hbar}x^2\psi = -\frac{2E}{\hbar\omega}\psi$$

define $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ (unitless) such that $x = \sqrt{\frac{\hbar}{m\omega}}\,\xi$ and $dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi$:

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$

$\left(K \equiv \frac{2E}{\hbar\omega}\right)$. So far, we have only recast our Schrodinger eqn in a more managable, unitless form. We haven't gotten any closer to solving it Tho!

Asymptotic behavior and "ansatz"

For large $\xi \gg 1$ (large x)

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2 \psi$$

This has approximate sol'n $\psi(\xi) = Ae^{-\xi^2/2} + \underbrace{\cancel{B}}_{=0 \text{ so that } \psi \text{ is normalizable.}} e^{+\xi^2/2}$

Check it!