Physics 374
Fall 2005
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MP 25 Solution

a. If we are ignoring gravity, there are only two forces on the ring. The first is the normal force coming from the contact between the ring and the rod. Since the rod is frictionless, that force must always be directly perpendicular to the rod, to the left. The other force on the ring is the tension force from the string. That force must be directed to the right, along the direction of the string. If the string makes a non-zero angle with the horizontal, then the normal and tension forces on the ring do not balance, and the ring will accelerate. Since the mass of the string is ~ 0, the acceleration will be very large (almost infinite). As a result, the ring will move very quickly, quickly establishing the zero-angle, no net force arrangement. Thus, the string tends to make a zero degree angle with the horizontal where it connects with the ring.

Thus our boundary conditions become:

$\frac{\partial y}{\partial x}$ = 0 at x = 0 and at x = L for all times

If we look for the normal modes of the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_o{}^2} \frac{\partial^2 y}{\partial t^2}$

$y(x,t) = f(x)\, e^{i\omega t}$

we get the equation for f

$\frac{d^2 f}{dx^2} = -\frac{\omega^2}{v_0{}^2} f$

which has the general solution

f(x) = Asin(kx) + Bcos(kx)

At x = 0 and L we have the boundary conditions

$\frac{\partial f}{\partial x} = 0$

At x = 0 this implies

$\frac{\partial f}{\partial x} = 0$ = Akcos(0) - Bksin(0) = Ak

so we must have A = 0.

That has simplified things quite a bit, but we still have one more boundary condition to apply...the one on the derivative of y at x = L:

$\frac{\partial f}{\partial x} =$ -Bk sin(kL)

so at x = L:

0 = sin(kL)

kL = nπ

$k_n = \frac{n\pi}{L}$ where n = 0,1,2,3 .... and $\omega_n = v_o k_n$

So we have

$y_n(x,t) = C_o \cos(\frac{n\pi x}{L}) \cos(\frac{n\pi\, v_o\, t}{L} + \phi)$

Note the difference here compared to the ends-tied-down example from class...here we include the n = 0 possibility. The reason is that here the solutions go as cos(nx), so putting in n = 0 doesn't zero out the y(x,t). When both ends were tied down, the solutions went as sin(nx), where putting n = 0 in would just give a superfluous zero everywhere.

Now, the main point of the whole find-the-set-of-normal-modes business is that it allows us to momentarily forget about the time dependence (i.e. the $\cos(\frac{n\pi\, v_o\, t}{L}+\phi)$ bit). With our set of normal modes, we'll be able to easily make a linear combination of our set of linearly independent solutions $f_n(x) = C_o \cos(\frac{n\pi x}{L})$ that gives us any initial shape we please. Once we have that initial shape, expressed in terms of a sum over the $f_n(x)$, we can simply tack on the time-dependence we found above, and have our complete solution.

Note that this approach is analogous to what we were doing before with our 2 coupled oscillators. There, we had two normal modes, out of which any motion could be built. Here, we have an infinite set of normal modes (which are functions now, not column vectors) out of which any motion can be built.

If we make our normal modes |n> $\to C_n \cos(\frac{n\pi x}{L})$, we should make sure to normalize them so we don't have extra constants floating around when we go to construct a certain initial shape.

$$1 = \langle n|n\rangle = \int_0^L C_n \cos(\tfrac{n\pi x}{L})\, C_n \cos(\tfrac{n\pi x}{L})\, dx = C_n{}^2 \int_0^L \cos^2(\tfrac{n\pi x}{L})\, dx = \tfrac{C_n{}^2 L}{2}$$

$$\text{so } C_n = \sqrt{\tfrac{2}{L}}$$

Thus, our normal modes are:

$$f_n(x) = \sqrt{\tfrac{2}{L}} \cos(\tfrac{n\pi x}{L}) \quad \text{where } n = 0,1,2,3...$$

As a final check, note our boundary conditions on $\frac{\partial y}{\partial x}$ at x = 0 and L are still satisfied by these normal modes.

b. Say we have a given initial shape, f(x) $\to$ |f>. Then we must be able to express this function as some linear combination of our |n> from above:

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle$$

To figure out what the expansion coefficients (i.e. the f_n) are, we can imagine hitting the equation above with a bra from the left:

$$\begin{aligned} \langle m|f\rangle &= \langle m| \sum_{n=0}^{\infty} f_n |n\rangle \\ &= \sum_{n=0}^{\infty} \langle m| f_n |n\rangle \\ &= \sum_{n=0}^{\infty} f_m \langle m |n\rangle \\ &= \sum_{n=0}^{\infty} f_m \delta_{m,n} \\ &= f_m \end{aligned}$$

Thus, the expansion coefficients are given by:

$$f_n = \langle n|f\rangle = \int_0^L \sqrt{\tfrac{2}{L}} \cos(\tfrac{n\pi x}{L})\, f(x)\, dx$$

c. I discussed above in part "a", why the n = 0 mode is mathematically possible and useful here but not in the previously seen ends-tied-down case. Let's take a closer look at what exactly goes on in this mode, although first it will be helpful to look at what happens with any other n ≠ 0 mode when we consider how it satisfies the wave equation.

$$y_n(x,t) = \sqrt{\tfrac{2}{L}} \cos(\tfrac{n\pi x}{L}) \cos(\tfrac{n\pi v_o t}{L} + \phi)$$

The long-winded math of part "a" was done in order to guarantee that this is a solution to

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_o{}^2} \frac{\partial^2 y}{\partial t^2}$$

Specifically:

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{n^2 \pi^2}{L^2}\right) \sqrt{\tfrac{2}{L}} \cos(\tfrac{n\pi x}{L}) \cos(\tfrac{n\pi v_o t}{L} + \phi)$$

$$\frac{\partial^2 y}{\partial t^2} = -\left(\frac{n^2 \pi^2 v_o{}^2}{L^2}\right) \sqrt{\tfrac{2}{L}} \cos(\tfrac{n\pi x}{L}) \cos(\tfrac{n\pi v_o t}{L} + \phi)$$

So we see that yes, indeed, the y_n(x,t)'s satisfy $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_o{}^2} \frac{\partial^2 y}{\partial t^2}$. There's a certain structure to the x-dependence of y_n(x,t) which brings out the $(\frac{n^2 \pi^2}{L^2})$ factor upon taking two derivatives. This MUST be matched by a similar structure in the time dependence of y_n(x,t), so that the double time derivative makes the right side of the wave equation match the left side.

The catch is what happens in the unique case of n = 0, where y_n(x,t) has NO x-structure. After all, $\sqrt{\frac{2}{L}} \cos(\frac{0\,\pi x}{L}) = \sqrt{\frac{2}{L}} \cos(0)$ = some non-zero constant. Now we have $\frac{\partial^2 y}{\partial x^2} = 0$, and the wave equation gives us some unique breathing room in determining the time dependence of our solution:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v_o{}^2} \frac{\partial^2 y}{\partial t^2}$$

$$0 = (\text{some constants}) \frac{\partial^2 y}{\partial t^2}$$

$$0 = \frac{\partial^2 y}{\partial t^2}$$

So apparently an n = 0 solution of the form $y_0(x,t) = C_1 + C_2 t$ is allowed. Look familiar? It's just the equation for the position of an object that is moving at a constant velocity C_2 that had an initial position C_1. Physically, this n = 0 mode can be excited by putting your hand under the entire length of string (so that all the pieces of the string move together) and shoving the whole string, together, upwards. Since the ends are free to move along the rods, the whole string will just translate upwards without any ripples or pulses bouncing back and forth on it.

Solution by T. Bing