Where does the $\sqrt{2\pi}$ in the Fourier transform come from?

To derive the factor we will start by assuming that

$$\langle x|k\rangle = Ae^{ikx}$$

and we will choose A so that when we use completeness to put a "k-basis" inside the delta function, it still works like it's supposed to. That is,

$$\delta(x-x') = \langle x|x'\rangle = \langle x|\left(\int_{-\infty}^{\infty}|k\rangle dk\langle k|\right)|x'\rangle = \int_{-\infty}^{\infty} dk\langle x|k\rangle\langle k|x'\rangle = A^2\int_{-\infty}^{\infty} dk\ e^{ik(x-x')}$$

We can't actually do the integral, so "works like it's supposed to" means, when we put it under an integral, it picks out the right value. In other words, if we integrate an arbitrary function $f(x)$, we get:

$$\int_{-\infty}^{\infty} dx'\ f(x')\ \delta(x-x') = f(x).$$

This means we must choose A such that

$$\int_{-\infty}^{\infty} dx'\ f(x')\ A^2\int_{-\infty}^{\infty} dk\ e^{ik(x-x')} = f(x)$$

The trick is, in order to evaluate the integral, we switch the order of integration and choose a particular function where we can do the integrals. So we want to evaluate

$$A^2\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx'\ e^{ik(x-x')}f(x') = f(x)$$

for a particular f. One where we can do the integrals explicitly is*

$$f(x) = e^{-x^2};$$

but to do the integrals explicitly, we have to evaluate two particular integrals. We'll do these as *Lemmas* (little theorems for use on the way to a bigger proof).

<u>Lemma 1</u>: $$\boxed{\int_{-\infty}^{\infty} dz\ e^{-z^2} = \sqrt{\pi}}$$

The proof of this is long, but elegant, and combines a variety of different tools.

Proof of Lemma 1:

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* For this proof, we are going to be mathematicians and assume everything is dimensionless. The proofs are messy enough without having to carry along constants to specify dimensions.

The idea is to generalize this integral from a number to a function by introducing a parameter. So we write:

$$\int_{-\infty}^{\infty} dz\, e^{-\lambda z^2} = I(\lambda)$$

If we cube this, we can write

$$\left(I(\lambda)\right)^3 = \int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} \int_{-\infty}^{\infty} dy\, e^{-\lambda y^2} \int_{-\infty}^{\infty} dz\, e^{-\lambda z^2} = \iiint dx\, dy\, dz\, e^{-\lambda\left(x^2+y^2+z^2\right)}$$

When we cubed it, we chose convenient (and suggestive) different dummy variable for each term and then combined them.

We can now treat the triple integral as if it were just a 3-D integral and go to polar coordinates. Since the integrand is only a function of $x^2 + y^2 + z^2 = r^2$, we can do the angular integrals and get:

$$\left(I(\lambda)\right)^3 = \iiint dx\, dy\, dz\, e^{-\lambda\left(x^2+y^2+z^2\right)} = \int_0^{\infty} 4\pi r^2\, dr\, e^{-\lambda r^2}$$

(We are replacing the sum over small cubes by sums over spherical shells whose surface area is $4\pi r^2$ and whose thickness is dr.) Now this integral is very much like $I(\lambda)$, but not quite. It only goes from 0 to infinity, not from –infinity to +infinity and it has an extra factor of r^2. We can easily handle this by using a couple of tricks. First, the integrand is symmetric so we can do the integral over the entire range and divide by two. Second, we can use a derivative trick.

$$\left(I(\lambda)\right)^3 = \iiint dx\, dy\, dz\, e^{-\lambda\left(x^2+y^2+z^2\right)} = \int_0^{\infty} 4\pi r^2\, dr\, e^{-\lambda r^2}$$

The first part is as follows:

$$\left(I(\lambda)\right)^3 = \int_0^{\infty} 4\pi r^2\, dr\, e^{-\lambda r^2} = \frac{1}{2}\int_{-\infty}^{\infty} 4\pi r^2\, dr\, e^{-\lambda r^2} = 2\pi \int_{-\infty}^{\infty} r^2\, dr\, e^{-\lambda r^2}$$

The second part works like this:

$$\left(I(\lambda)\right)^3 = 2\pi \int_{-\infty}^{\infty} r^2\, dr\, e^{-\lambda r^2} = 2\pi\left(-\frac{d}{d\lambda}\int_{-\infty}^{\infty} dr\, e^{-\lambda r^2}\right)$$

When the derivative with respect to lambda is taken inside the integral, it brings down the required factor of $-r^2$, but we can do the integral first and differentiate second.

Well, we can't actually do the integral (yet), but we can recognize it as $I(\lambda)$ again, just with a different dummy variable (r this time). So we get the interesting equation:

$$\left(I(\lambda)\right)^3 = -2\pi \frac{d}{d\lambda} I(\lambda)$$

We can solve this equation by standard techniques!

$$\frac{dI}{I^3} = -\frac{d\lambda}{2\pi}$$
$$-\frac{1}{2I^2} = -\frac{\lambda}{2\pi} + C$$

where C is an integration constant. We can figure it out by looking at our integral. If λ goes to 0, then the integrand becomes 1 and the integral goes to infinity. Putting these values in, we see that C has to be 0. We therefore have the result:

$$\left(I(\lambda)\right)^2 = \frac{\pi}{\lambda}$$

Since we want the value for λ = 1, we just get the quoted result.

QED (Lemma 1)

<u>Lemma 2</u>: $\boxed{\int_{-\infty}^{\infty} dz\, e^{-z^2} e^{ibz} = \sqrt{\pi} e^{-b^2/4}}$

Proof of Lemma 2.

We do this one by completing the square in the exponential and changing variables thus:

$$-z^2 + ibz = -\left(z^2 - ibz\right) = -\left(z^2 - ibz - \frac{b^2}{4}\right) - \frac{b^2}{4} = -\left(z - \frac{ib}{2}\right)^2 - \frac{b^2}{4}$$

Defining a new integration variable, $z' = z - \frac{ib}{2}$ and noting that, since the shift is just a constant, $dz' = dz$, we can write:

$$\int_{-\infty}^{\infty} dz\, e^{-z^2} e^{ibz} = \int_{-\infty}^{\infty} dz'\, e^{-z'^2} e^{-b^2/4} = \sqrt{\pi} e^{-b^2/4}$$

where we have used Lemma 1 to evaluate the z' integral.

QED (Lemma 2)

Now we are ready to derive the factor. If the completeness relation inside the delta function is to work we must have

$$\int_{-\infty}^{\infty} dx'\, f(x')\, A^2 \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = f(x)$$

for any function. In particular, it must work for the function e^{-x^2}

Putting this in and applying Lemma 2 gives

$$A^2 \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx'\, e^{ik(x-x')} e^{-x'^2} = A^2 \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, e^{-ikx'} e^{-x'^2} = A^2 \int_{-\infty}^{\infty} dk\, e^{ikx}\, \sqrt{\pi} e^{-k^2/4} .$$

Applying Lemma 2 again gives

$$A^2 \int_{-\infty}^{\infty} dk\, e^{ikx} \sqrt{\pi} e^{-k^2/4} = A^2 \sqrt{\pi} \int_{-\infty}^{\infty} 2dw\, e^{i2wx}\, e^{-w^2} = A^2 2\pi e^{-(2x)^2/4} = A^2 2\pi e^{-x^2}$$

Since this has to equal e^{-x^2} we must require $1 = A^2 2\pi$ or $A = \frac{1}{\sqrt{2\pi}}$.

E. F. Redish
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