Complex Numbers

Motivation

Complex numbers are a way to combine the idea of number (addition, multiplication, distribution, etc.) with the idea of vectors in the plane, producing a powerful tool.

The power is provided mathematically by the fundamental theorem of algebra. Recall that this theorem says that any algebraic polynomial equation of the n^{th} degree — something of the form

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

always has exactly n complex roots; that is, it can be factored into the form

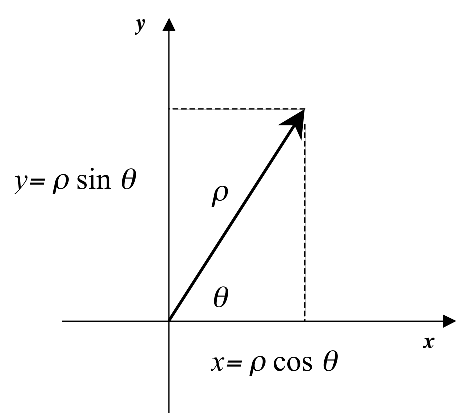
$$(z - z_1)(z - z_2)\ldots(z - z_n) = 0$$

where the z_i are (possibly complex) constants, the solutions of the equation. We will see this give us tremendous power for figuring out solutions of ordinary linear (i.e., the unknown comes in as the first power in every term) differential equations with constant coefficients.

The power is provided physically whenever we have a physical system in which there are two things (numbers or functions) that are related in the way complex numbers are. One such case is for oscillations of objects governed by Newton's laws. Since these are second order differential equations, sometimes linear, sometimes with constant coefficients as is our prototype oscillator example, the SHO) the two solutions look like sines and cosines. These are related in the way that the parts of a complex function are.

Definitions and Polar Coordinates

The basic definition of complex numbers is that we extend our real number line by adding a number not on that line, the square root of –1 and all multiples of that by a real number. This leads to two independent (orthogonal) real lines, equivalent to a plane. We write $i = \sqrt{-1}$ and any complex number as $z = x + iy$ where x and y are real numbers. These are represented on a plane as shown in the figure on the right. The complex number z is represented by the vector.



Polar coordinates are very convenient when working with complex numbers. We define the length of the vector z and the angle it makes with the x axis as:

$$\rho = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

This gives the inverse relations

$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$

This gives the important representation of a complex number

$$z = \rho\left(\cos\theta + i\sin\theta\right)$$

The de Moivre Theorem

The representation shown above appears very frequently so it pays us to look at what it means. We have worked out power series representations of cos and sin so we can put them together and see what we have.

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \ldots$$
$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \ldots$$

Putting these together as $\cos\theta + i\ \sin\theta$ gives

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 \ldots$$

This looks quite systematic, There is a term in every power with a nice systematic set of coefficients — except for the i's. If, however, we notice that $i^2 = -1$ we can note that replacing the minus signs by the square of i gives a nice result.

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \ldots$$
$$\cos\theta + i\sin\theta = e^{i\theta}$$

The is take care of all the signs and just give us the familiar power series for the exponential. This now gives us the nice representation for a complex number:

$$z = x + iy = \rho e^{i\theta}$$

The polar representation of the complex number is particularly nice because it's easy to divide or multiply that way.

$$z_1 = x_1 + iy_1 = \rho_1 e^{i\theta_1}$$
$$z_2 = x_2 + iy_2 = \rho_2 e^{i\theta_2}$$
$$z_1 z_2 = \rho_1 e^{i\theta_1} \rho_2 e^{i\theta_2} = (\rho_1 \rho_2) e^{i(\theta_1 + \theta_2)}$$

You just multiple the lengths and add the angles.

Exercise: To be sure this is consistent, try showing that if you multiply the x-y forms of the complex numbers then the results give the length and angle for the product that the polar forms give.

The polar result in exponential form gives a lot of nice and remarkable results.

$$e^{i\pi} + 1 = 0$$
$$\text{If } z = \rho(\cos\theta + i\sin\theta) \text{ then}$$
$$z^n = \rho^n(\cos n\theta + i\sin n\theta)$$

The first of these is called Euler's theorem. It combines e, i, π, 1, and 0. Many consider this the most elegant of all mathematical equations. The second is called de Moivre's theorem. If you raise the polar form of z to some power and equate it to the de Moivre form, you can easily get expressions for cos nθ and sin nθ in terms of cos θ and sin θ.

For example:

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$$

Equating the real part on the left to the real part on the right and the imaginary part on the left to the imaginary part on the right gives the well-known (but otherwise cumbersome to prove) results:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

If you try to prove these using trig identities you will see how much work de Moivre saves us!

Exercise: Use de Moivre's theorem to figure out what cos 3θ and sin 3θ are in terms of sin θ and cos θ.