Dirac Notation

Motivation

Notation can help us substantially in thinking about and manipulating symbolic representations meant to describe complex physical phenomena. The brain's working memory can only manipulate a small number of ideas at once ("7±2"). We handle complex ideas by "chunking" — binding together many things and manipulating them as a single object. Another way we extend our range is by storing information outside of our brains temporarily and manipulating external objects or symbols, like an abacus or equations written on a piece of paper.

Notation — the way we choose to organize our symbology to represent something — can play a powerful role in helping us think about a complex situation. In Maxwell's day, the equations for electric and magnetic fields were written out component by component, so his equations took up a full page of text. Looking at those equations, it's clear that there is a regularity to the equations that should allow for some compression. When Gibbs introduced his vector notation, Maxwell's equations could be collapsed into 4 lines. Furthermore, they had the advantage that they did not depend on the choice of coordinate system. You could use the same equations, manipulate them as you wished, and then introduce a particular choice of coordinate (e.g., a particular orientation of rectangular coordinates or a convenient set of curvilinear coordinates) after you were done.

A similar situation pertains for dealing with linear spaces. In some cases, we might want to describe a system of coupled oscillators with the coordinates of the masses. In other cases, we might want to describe them in terms of how much of each normal mode is excited. This change corresponds to a change of coordinates in the linear space describing the state of the system. We would like to have a representation that describes the state without specifying the particular coordinates used to describe them.

Other cases where linear spaces are useful include cases where complex numbers are helpful in describing the physical system. Some examples of this include: polarization of electromagnetic waves (linear vs. circular), wave motion of mechanical systems (Fourier analysis), and quantum physics.

The Dirac notation for states in a linear space is a way of representing a state in a linear space in a way that is free of the choice of coordinate but allows us to insert a particular choice of coordinates easily and to convert from one choice of coordinates to another conveniently. Furthermore, it is oriented in a way (bra vs. ket) that allows us to keep track of whether we need to take complex conjugates or not. This is particularly useful if we are in an inner-product space. To take the length of a complex vector, we have to multiply the vector by its complex conjugate — otherwise we wont get a positive number. The orientation of the Dirac representation allows us to nicely represent the inner product in a way that keeps careful track of complex conjugation.

The Dirac Representation of States in a Inner Product Space

Suppose we consider a two-dimensional complex linear inner-product space. A general vector in this space takes the form

$$\vec{a} = \alpha \vec{e}_1 + \beta \vec{e}_2$$

where α and β are complex numbers and e_1 and e_2 are (real) basis vectors. We define our inner product to be

$$\vec{a} = \alpha \vec{e}_1 + \beta \vec{e}_2$$
$$\vec{b} = \gamma \vec{e}_1 + \delta \vec{e}_2$$
$$\vec{a} \cdot \vec{b} = \alpha^* \beta + \gamma^* \delta$$

We put complex conjugates on the left vector's components so that

$$\vec{a} \cdot \vec{a} = \alpha^* \alpha + \beta^* \beta = |\alpha|^2 + |\beta|^2 \neq \alpha^2 + \beta^2$$

We use the complex conjugate because if we just took $\alpha^2 + \beta^2$, it wouldn't always be positive. It wouldn't necessarily even be a real number and we want the length of a vector to be a real positive number.

This is very natural if we are working in a particular coordinate basis so we can write the vector as a two-component (complex) vector. The dot product is then just the matrix product of a row vector with a column vector:

$$\vec{a} \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \vec{a}^* \leftrightarrow \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

$$\vec{a} \cdot \vec{a} \rightarrow \vec{a}^* \cdot \vec{a} \leftrightarrow \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^* \alpha + \beta^* \beta$$

Notice that in a complex space, all complex coefficients are in the space, so given the vector $\vec{a} \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ the vector $\vec{a}^* \leftrightarrow \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}$ is just another vector in the space. But for the vector $\vec{a}$ it is a special vector. It is associated with $\vec{a}$ by the operation of complex conjugation. To get the length of a vector in a complex space, instead of taking the dot product of the vector with itself, we take the dot product of the vector with another vector in the space — the one associated with the original vector by complex conjugation.

Now using the one- and two-column representations of our vectors is fine if we are never going to be changing coordinates. If we are, the column vector becomes ambiguous. Which basis vectors do we mean that a particular column goes with? We want a representation that is basis independent and allows us to put a particular basis in as we chooses. We also want one that will keep track of whether we are talking of our original vector or whether we have complex conjugated it in order to take an inner product.

A notation that does this very nicely was invented by the physicist P. A. M. Dirac for quantum physics — but we can use it anywhere. The notation chooses to enclose the vector symbol in a surround marker rather than putting an arrow over it. Dirac chose the

notation of "half a bracket" (a *ket*) to represent a vector. The other half of the bracket (a *bra*) was used to represent the vector's complex conjugate. Putting them together gave a "bra-ket" or "bracket" that represented a number — the inner product. Here's how it works in symbols:

$$\vec{a} = |a\rangle \quad \vec{a}^* = \langle a| \quad \vec{a}^* \cdot \vec{a} = \langle a\|a\rangle = \langle a|a\rangle$$

Notice when a bra and a ket are put together to make a number the two lines are collapsed into a single line to show that they are bound into a single object. In a particular basis, this corresponds to the component notation as follows.

$$\langle a|b\rangle = \vec{a} \cdot \vec{b} = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1^* b_1 + a_2^* b_2$$

The Outer Product

The Dirac notation nicely represents something else that happens in a linear space: matrices. A matrix is a linear operator that acts on a vector to give another vector back. The fact that it is linear means:

$$\text{If} \quad \mathbf{A}\vec{a} = \vec{a}' \quad \text{and} \quad \mathbf{A}\vec{b} = \vec{b}'$$

$$\text{then} \quad \mathbf{A}(\alpha\vec{a} + \beta\vec{b}) = \alpha\left(\vec{a}' + \vec{b}'\right)^.$$

In a way, the simplest way to do this is to take the original vector, take its dot product with a second vector to get a number, and multiply is by a third vector. This combination of the second and third vector is called the *outer product* of the vectors, written $\vec{a} \otimes \vec{b}$. This is a linear operator acting on a vector by the rule

$$\left(\vec{a} \otimes \vec{b}\right)\vec{c} = \vec{a}\left(\vec{b} \cdot \vec{c}\right)$$

where the combination in the parentheses on the right is the dot product. This therefore takes a vector in the $\vec{c}$ direction and gives back one in the $\vec{a}$ direction.

For a 2x2 example in component notation, this is just the reverse of the inner product; where the inner product was a row vector (1x2) times a column vector (2x1) yielding a number (1x1), the outer product is a column vector (2x1) times a row vector (1x2) yielding a matrix (2x2).

$$\vec{a} \otimes \vec{b} \leftrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} b_1^* & b_2^* \end{pmatrix} = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{pmatrix}$$

Remember the general rule that you can multiply an NxK matrix times a KxM matrix to get an NxM matrix. In equations, the rule is

$$\left(\mathbf{AB}\right)_{ij} = \sum_{k=1}^{K} A_{ik} B_{kj} \qquad i = 1...N, j = 1...M.$$

To see what is really going on here, it is useful to look at what happens when we take the outer product of two basis vectors. If we have an orthonormal basis, $\hat{e}_i \ i = 1...N$, then the

outer product $\hat{e}_i \otimes \hat{e}_j$ acting on a vector $\vec{a}$ picks out the j-th component of the vector and gives it back in the i-th direction; that is:

$$\left(\hat{e}_i \otimes \hat{e}_j\right)\vec{a} = \hat{e}_i\left(\hat{e}_j \cdot \vec{a}\right) = a_j\hat{e}_i$$

The matrix that does this job is the one with 1 in the j-th row and i-th column and 0 everywhere else. We can therefore write any matrix as an outer product operator

$$\mathbf{A} = \sum_{i,j=1}^{N} A_{ij}\hat{e}_i \otimes \hat{e}_j$$

The Dirac Representation of the Outer Product

The outer product of two vectors in an inner product space (i.e., a linear space where an inner product is defined) is particularly convenient and conveys a lot of meaning, especially in a space where you can take complex conjugates. The Dirac form of the outer product is like the row/column vector notation except without reference to a particular basis; it is the opposite of the inner product. We write:

$$\vec{a} \otimes \vec{b} = |a\rangle\langle b|$$

The vector on the right goes in the bra since we know its rule is that it is going to come up on a vector and inner product with it from the left. We also remember the vector on the left in an inner product has to have its complex conjugate taken. The vector on the left in an outer product goes in the ket since its rule tells us this is the direction of the state vector we are going to be left with and we consider the typical state vector to be a ket. In the outer product form with the circled "x", we just have to remember which vector gets the complex conjugate. The Dirac notation reminds us by its structure.

You can quickly get used to the idea of how this works if you think of a bra-ket (the inner product) as a bracket — the two vectors are "striving" to collapse comfortably into a number. Whereas the ket-bra (the outer product) is not a bracket — it is open — "hungry" to devour a ket on the right and collapse back into a vector. The orthography (the way the symbols are structured) makes this quite natural:

$$\left(|a\rangle\langle b|\right)|c\rangle = |a\rangle\left(\langle b|c\rangle\right).$$

(Read that carefully: On the left, we have an outer product acting on a vector. Removing the parentheses just puts a bra and a ket together. We collapse them into a bra-ket — an inner product or a number.)

The rule about the matrices and the outer product produces a pair of useful results.

$$\mathbf{A} = \sum_{i,j=1}^{N} A_{ij}|e_i\rangle\langle e_j| = \sum_{i,j=1}^{N} |e_i\rangle A_{ij}\langle e_j|$$

$$\mathbf{I} = \sum_{i,j=1}^{N} |e_i\rangle\delta_{ij}\langle e_j| = \sum_{i=1}^{N} |e_i\rangle\langle e_i|$$