HW#5 —Phys776—Spring2005 Prof. Ted Jacobson
Due Thursday, April 14, before class. Room 4115, (301)405-6020
jacobson@physics.umd.edu

1. Consider a three-dimensional spherical box of circumferential radius R, that is held at temperature T (as measured by a static observer on the box). Show that:

   (a) A black hole inside can be in equilibrium with the box (i.e. the Hawking temperature measured at the box is equal to T) only if $T > 3\sqrt{3}/8\pi R$, in which case there are two equilibrium values for the mass $M(T, R)$.

   (b) The specific heat for the larger (smaller) $M(T, R)$ is positive (negative). Hence the larger black hole is (locally, at least) in stable equilibrium. Show that its mass M always satisfies $R < 3M$.

2. Consider a (three-dimensional) box containing radiation and possibly a black hole in the microcanonical ensemble with total energy E. If the box is sufficiently large, the most entropic configuration will consist of pure radiation, spread out in the box. If the box is sufficiently small, the most entropic configuration will contain a black hole of some mass M.

   (a) Show that the configuration containing a black hole is a local maximum of the entropy if $E_r < M/4$. (Neglect the effect of the black hole on the radiation.)

   (b) Show that the configuration containing a black hole is a local maximum of the entropy if and only if $V < aE^5$ (in Planck units), where V is the volume of the box and a is a constant which you need not compute.

   (c) In a box with the borderline volume $V = aE^5$, compare the entropy of pure radiation with energy E to the entropy of radiation of energy $E_r = E/5$ together with a black hole of mass $M = 4E_r$. (If I'm not mistaken, the ratio is $2/5^{1/4}$, which is greater than unity. Hence the black hole is not the most likely configuration, although for a box that is a bit smaller it becomes so...)

3. The line element of a Schwarzschild-anti-de Sitter black hole spacetime in five dimensions is

$$ds^2 = f(r)\, dt^2 - f(r)^{-1}\, dr^2 - r^2\, d\Omega_3^2, \tag{1}$$

   with

$$f(r) = \frac{r^2}{R^2} + 1 - \frac{r_0^2}{r^2}, \tag{2}$$

   where R is the curvature radius of the AdS background to which the black hole metric asymptotes.

   (a) Find the surface gravity of this black hole as a function of the horizon radius r_+ where $f(r_+) = 0$. Since the Killing field diverges at infinity, it cannot be normalized there, and the surface gravity is ambiguous until one specifies the normalization. For the normalization, choose the Killing field equal to $\partial/\partial t$ in the above coordinates.

   (b) Find the minimum horizon radius (in terms of R) for the black hole to have positive specific heat, and find the temperature of this critical black hole.