HW#3 —Phys776—Spring2005
Due Tuesday, March 8, before class.

Prof. Ted Jacobson
Room 4115, (301)405-6020
jacobson@physics.umd.edu

1. *Raychaudhuri equation & cosmology*

   The Raychaudhuri equation for a timelike geodesic congruence parameterized by proper time t is
   $$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b, \tag{1}$$
   where θ, σ_{ab}, and ω_{ab} are the expansion, shear, and twist, respectively, and u^a is the 4-velocity. Consider a a homogeneous isotropic universe (i.e. a Robertson-Walker spacetime, possibly with spatial curvature) with scale factor $a(t)$, comoving energy density ρ, and pressure p.

   (a) Use the Raychaudhuri equation for the cosmological geodesic flow (u^a orthogonal to the homogeneous isotropic surfaces) to derive the second order Friedmann equation, $\ddot{a}/a = -(4\pi/3)(\rho + 3p)$. Use the Einstein equation only to evaluate $R_{ab}u^a u^b$ in terms of ρ and p.
   [Note that a positive pressure adds to the attraction of energy density, while a negative pressure substracts. If $p < -\rho/3$ the universe is accelerating. Vacuum energy is locally Lorentz invariant, hence has $T_{ab} = \rho_{\rm vac} g_{ab}$, so $p = -\rho_{\rm vac}$. It can therefore produce acceleration.]

   (b) Use the Raychaudhuri equation to show that, if $\rho + 3p > 0$, a ball of test particles initially at rest with respect to each other, and whose center is at rest with respect to the cosmological fluid, will *contract* rather than expand. Explain how this is consistent with the cosmological expansion.

2. *Hypersurface orthogonal vector fields*

   A vector field V^a that is orthogonal to a foliation by hypersurfaces is called *hypersurface orthogonal*. The hypersurfaces can be characterized as the level sets of some function S. Since V_a and $\nabla_a S$ both vanish when contracted with any vector tangent to the constant S hypersurfaces, they must be proportional, i.e. $V_a = f\nabla_a S$ for some function f.

   (a) Show that if V^a has this form then
   $$\nabla_{[a}V_{b]} = V_{[a}W_{b]} \tag{2}$$
   for some co-vector W_b. (The converse, which I do not ask you to prove, is *Frobenius' theorem*.)

   (b) Show that the condition (2) holds if and only if
   $$V_{[a}\nabla_b V_{c]} = 0. \tag{3}$$
   (*Hint for the if part*: Expand out (3) and contract on the index c with any vector X^c such that $X^c V_c \neq 0$.)

(c) Show that in Minkowski spacetime the Killing fields ∂_t and ∂_ϕ are hypersurface orthogonal, but $\partial_t + \Omega\partial_\phi$ (where Ω is constant) is *not*.

(d) Show that in the Kerr spacetime, neither of the Killing vector fields ∂_t nor ∂_ϕ (in B-L coordinates) are hypersurface orthogonal.

[A *stationary* spacetime is one with a timelike Killing vector. If the timelike Killing vector is also "hypersurface orthogonal", the spacetime is called *static*. The Kerr spacetime is stationary, but not static.]

(e) Since $\chi = \partial_t + \Omega_H\partial_\phi$ is normal to the Kerr horizon, it is hypersurface orthogonal *on the horizon* (though not elsewhere). Use this property to show that the square of the surface gravity is given by

$$\kappa^2 = -\frac{1}{2}(\nabla_a\chi_b)(\nabla^a\chi^b). \tag{4}$$

For this purpose use the relation $\chi^a\nabla_a\chi_b = \kappa\chi_b$, which holds on the horizon, to define the surface gravity κ.

(f) Show that if u^a is hypersurface orthogonal, and is a timelike or spacelike, affinely parametrized geodesic vector field, then the twist vanishes, i.e. $\omega_{ab} := \nabla_{[a}u_{b]} = 0$.

(g) Show that if k^a is hypersurface orthogonal, and is a null, affinely parametrized geodesic vector field, then the twist has the form $\omega_{ab} := \nabla_{[a}u_{b]} = k_{[a}v_{b]}$, where v^a is orthogonal to k^a. Conclude that $\omega_{ab}\,\omega^{ab} = 0$.