

Index notation

Vector notation like $\mathbf{E}$ or $\vec{E}$ is compact and convenient in many ways, but sometimes it is clumsy and limiting. Some relations are difficult to see, prove, or even to write. On the other hand, writing out the three components of a vector is even clumsier. A good compromise is to indicate the components by an index that runs from 1 to 3, denoting the different components: E^i , $i = 1, 2, 3$. In a more complete treatment of these matters, the index can refer to components in any kind of coordinate system. In that setting, superscript indices and subscript indices refer to different things. Here I will keep things simple by restricting to Cartesian coordinates, so that the index can be placed either as a superscript or a subscript, and I'll move them up and down according to notational convenience.

Tensors

Two vectors can be multiplied component by component to obtain an object with nine independent components. For example we can write $E^i E^j$, the independent components of which are $E^1 E^1$, $E^1 E^2$, $E^1 E^3$, $E^2 E^1$, etc. This is called a tensor of rank two. This particular tensor is said to be “symmetric”, since the ij component is equal to the ji component, $E^i E^j = E^j E^i$. If A^i and B^j are two non-parallel vectors, then $A^i B^j \neq A^j B^i$, so $A^i B^j$ is not symmetric. The order of writing the components is arbitrary, however, since they are just numbers: $A^i B^j = B^j A^i$. This product of A^i and B^j is called the “outer product”, or “direct product”, or, perhaps more properly, “tensor product”. We can also form higher rank tensors, like $A^i B^j C^k$, and we can add tensors, like $A^i B^j + C^i D^j$. In fact, the most general rank two tensor would be obtained by a linear combination of the nine tensor products you can form from a pair of vectors in a given basis.

Electric field of a dipole

As an example, we compute the electric field of a dipole potential $V(\mathbf{r}) = k\mathbf{p} \cdot \hat{\mathbf{r}}/r^2 = k\mathbf{p} \cdot \mathbf{r}/r^3$ (the latter form is more convenient in this context). This one is easier to compute in spherical coordinates, $V = kp \cos \theta / r^2$, using the expression for the gradient in spherical coordinates, which immediately yields

$$\mathbf{E} = kpr^{-3}(2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}). \quad (1)$$

To see how things work, and to get some practice, let's do it instead in Cartesian coordinates using the index notation. Then

$$V(\mathbf{r}) = kp^i r^i r^{-3}, \quad (2)$$

where the summation over the three values of the repeated index i is understood. This *Einstein summation convention* makes writing things out much simpler. To keep it unambiguous we just have to agree never to use the same index twice, unless it's being summed with another index. Note that if both i 's are replaced by, say, j 's in the above expression, nothing changes. That is, the dummy index i can have any name.

Let's now evaluate the electric field $E^i = -\partial^i V$. The first step, since I used the index i for derivative, is to change the name of the index we sum over in V to something else, say j :

$$E^i = -\partial^i V \quad (3)$$

$$= -\partial^i (kp^j r^j r^{-3}). \quad (4)$$

Next, since k and p^j are constants, and the derivative of a sum is the sum of the derivatives, we may continue as

$$= -kp^j \partial^i (r^j r^{-3}). \quad (5)$$

Now for each value of i and j we have the derivative of a product, on which we may use the product rule:

$$= -kp^j [(\partial^i r^j) r^{-3} + r^j \partial^i r^{-3}]. \quad (6)$$

In the first term, $\partial^i r^j = \delta^{ij}$, because ∂^i is the partial derivative with respect to the i th coordinate, and r^j is nothing but the j th coordinate (and δ^{ij} is the Kronecker delta). In the second term we can use the chain rule to write $\partial^i r^{-3} = -3r^{-4} \partial^i r = -3r^{-4} \hat{r}^i$ (using $\partial^i r = \hat{r}^i$). Hence the next steps are

$$= -kp^j (\delta^{ij} r^{-3} - 3r^j \hat{r}^i r^{-4}) \quad (7)$$

$$= -kr^{-3} p^j (\delta^{ij} - 3\hat{r}^i \hat{r}^j). \quad (8)$$

Now we still have the original sum over j values. In the first term this just gives us p^i , and in the second term it yields $(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{r}^i$. Hence, finally, converting back to vector notation,

$$\mathbf{E} = kr^{-3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] \quad (9)$$

You should be able to show that this is identical to the result (1) found using spherical coordinates.

Force on a dipole

As another example, let's consider the expression we derived for the force on a dipole in an electric field,

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}. \quad (10)$$

Using index notation this looks like

$$F^i = p^j \partial^j E^i. \quad (11)$$

Since the electric field is the gradient of a scalar, $E^i = -\partial^i V$, and since partial derivatives commute, $\partial^i \partial^j = \partial^j \partial^i$, we have

$$\partial^i E^j = -\partial^i \partial^j V = -\partial^j \partial^i V = \partial^j E^i. \quad (12)$$

This amounts to the same as the fact that the curl of a gradient always vanishes. Using it in (36) yields

$$F^i = p^j \partial^i E^j. \quad (13)$$

Then, if the dipole moment p^j is a constant, we can move it inside the derivative,

$$F^i = \partial^i p^j E^j \quad (14)$$

or, in index-free notation,

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}). \quad (15)$$

later in the course we'll encounter examples where this index notation is really much more convenient than any alternative I know of.

Laplace's equation, zero divergence and zero curl

Laplace's equation:

$$\partial_i \partial_j V = 0. \quad (16)$$

An electrostatic or magnetostatic field in vacuum has zero curl, so is the gradient of a scalar, and has zero divergence, so that scalar satisfies Laplace's equation. But also the electric field vector itself satisfies Laplace's equation, in that each component does. Here is an index proof:

$$\partial_i \partial_i E_j = \partial_i \partial_j E_i = \partial_j \partial_i E_i = 0. \quad (17)$$

The first equality follows from the curl-free equation (or the fact that $E_i = -\partial_i V$), the second from commuting partials, and the last from the divergence-free condition.

Another interesting result is that the square of the magnitude of the electric field has non-negative Laplacian, which means that it can have a local minimum, but never a local maximum. Here's a proof:

$$\partial_i \partial_i (E_j E_j) = 2 \partial_i (E_j \partial_i E_j) \quad (18)$$

$$= 2(\partial_i E_j)(\partial_i E_j) + 2E_j \partial_i \partial_i E_j \quad (19)$$

$$= 2(\partial_i E_j)(\partial_i E_j) \quad (20)$$

$$\geq 0 \quad (21)$$

The second to last line follows since the electric field satisfies Laplace's equation, and the positivity in the last line is just because the expression is a sum of squares, each of which is non-negative.

Since the field strength can have no local maximum in vacuum, electric discharge will not be initiated in vacuum, but rather at a non-zero charge density. Also an electric dipole or paramagnetic dipole cannot be in a stable equilibrium, although a diamagnetic dipole can be. This is explained in the article "Of flying frogs and levitrons", MV Berry and AK Geim, linked at the Notes page of the course.

Magnetic dipole moment

Here I'll explain why the magnetic dipole moment can be written as

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{J} d\tau. \quad (22)$$

From the product rule we have:

$$\partial_k(x^i x^j J^k) = x^i J^j + x^j J^i + x^i x^j \partial_k J^k, \quad (23)$$

where I've used $\partial_k x^i = \delta_k^i$, and the implicit sum over k thus replaces the k index on J^k by the index i , etc. The charge continuity equation implies the last term is equal to $-\dot{\rho}$, and the left hand side is a total divergence, hence vanishes when integrated over a region whose boundary has no current, so integrating this equation yields an integral identity,

$$\int (x^i J^j + x^j J^i) d\tau = \int \dot{\rho} x^i x^j d\tau. \quad (24)$$

Now consider the expression that occurs in the dipole term of the vector potential:

$$(\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}' = (\mathbf{r}' \times \mathbf{J}') \times \hat{\mathbf{r}} + (\hat{\mathbf{r}} \cdot \mathbf{J}') \mathbf{r}' \quad (25)$$

$$= (\mathbf{r}' \times \mathbf{J}') \times \hat{\mathbf{r}} + [(\hat{\mathbf{r}} \cdot \mathbf{J}') \mathbf{r}' + (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}'] - (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}', \quad (26)$$

hence

$$(\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}' = \frac{1}{2}(\mathbf{r}' \times \mathbf{J}') \times \hat{\mathbf{r}} + \frac{1}{2}[(\hat{\mathbf{r}} \cdot \mathbf{J}') \mathbf{r}' + (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}']. \quad (27)$$

If this is integrated over all space we obtain, using the identity (24),

$$\int (\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}' d\tau' = \mathbf{m} \times \hat{\mathbf{r}} + \frac{1}{2} \mathbf{Q} \cdot \hat{\mathbf{r}}, \quad (28)$$

where the magnetic moment $\mathbf{m}$ is given by (22), and $\mathbf{Q}$ is the “second moment” of the charge distribution,

$$\mathcal{Q}^{ij} = \int \rho x^i x^j d\tau. \quad (29)$$

The index notation for $\mathbf{Q} \cdot \hat{\mathbf{r}}$ is $\mathcal{Q}^{ij} \cdot \hat{r}^j$. The trace-free part $Q^{ij} = \mathcal{Q}^{ij} - \frac{1}{3} \mathcal{Q}^{kk} \delta^{ij}$ is the quadrupole moment tensor.

Torque and force on a magnetic dipole

The magnetic force on a current element is

$$d\mathbf{F} = dq \mathbf{v} \times \mathbf{B} = \mathbf{J} \times \mathbf{B} d\tau, \quad (30)$$

since $dq \mathbf{v} = (\rho d\tau) \mathbf{v} = \mathbf{J} d\tau$.

Torque on a current distribution

The torque on a current distribution in a constant magnetic field is thus

$$\mathbf{N} = \int \mathbf{r} \times d\mathbf{F} \quad (31)$$

$$= \int \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) d\tau \quad (32)$$

$$= \int [(\mathbf{r} \cdot \mathbf{B}) \mathbf{J} - (\mathbf{r} \cdot \mathbf{J}) \mathbf{B}] d\tau \quad (33)$$

$$= \mathbf{m} \times \mathbf{B}, \quad (34)$$

where in the last step the first term has the same form as (28), and the second term vanishes by virtue of (24) when you set $i = j$ and sum over i .

Force on a current distribution

The force on a current distribution is

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} d\tau. \quad (35)$$

If $\mathbf{B}$ is constant, it comes out of the integral, and since $\int \mathbf{J} d\tau = 0$ the total force vanishes. If $\mathbf{B}$ is not constant, we can expand it in a Taylor series, $\mathbf{B} = \mathbf{B}_0 + x^l (\partial_l \mathbf{B})_0 + \dots$, where I've assumed $x^l = 0$ lies inside the current distribution. If the current is spread out over a small enough region, it makes sense to drop the higher order terms in the Taylor expansion. In this approximation we have

$$\mathbf{F} = \left(\int x^l \mathbf{J} d\tau \right) \times (\partial_l \mathbf{B})_0. \quad (36)$$

At this point, I can't see any way to proceed other than to introduce the alternating symbol.

The alternating symbol ϵ^{ijk}

The cross product can be written with the help of the alternating symbol ϵ^{ijk} , which is equal to 1 or -1 when ijk is an even or odd permutation of 123 respectively, and zero otherwise. Thus,

$$(\mathbf{V} \times \mathbf{W})^i = \epsilon^{ijk} V^j W^k. \quad (37)$$

Conversely,

$$V^j W^k - V^k W^j = \epsilon^{jkl} (\mathbf{V} \times \mathbf{W})^l. \quad (38)$$

This can be used to express the magnetic dipole moment in a useful form:

$$\int x^j J^k d\tau = \frac{1}{2} \int (x^j J^k - x^k J^j) d\tau; \quad (39)$$

$$= \frac{1}{2} \epsilon^{jkl} \int (\mathbf{x} \times \mathbf{J})^l \quad (40)$$

$$= \epsilon^{jkl} m^l. \quad (41)$$

A very useful identity, which underlies most vector identities, is that if two alternating symbols are multiplied, and one index is summed, the result is an antisymmetric combination of Kronecker deltas:

$$\epsilon^{ijk} \epsilon^{mnk} = \delta^{im} \delta^{jn} - \delta^{in} \delta^{jm} \quad (42)$$

(This can be proved by just checking each possibility for the free (un-summed) indices $ijmn$.)

Force on a current distribution (*continued*)

Using these results involving the alternating symbol, the force on a dipole (36) can now be written as

$$\mathbf{F}^i = \epsilon^{ijk} \left(\int x^l \mathbf{J}^j d\tau \right) (\partial_l \mathbf{B}^k)_0 \quad (43)$$

$$= \epsilon^{ijk} \epsilon^{ljs} m^s (\partial_l \mathbf{B}^k)_0 \quad (44)$$

$$= (\delta^{il} \delta^{ks} - \delta^{is} \delta^{kl}) m^s (\partial_l \mathbf{B}^k)_0 \quad (45)$$

$$= m^k (\partial^i \mathbf{B}^k)_0, \quad (46)$$

where in the last step I used $\nabla \cdot \mathbf{B} = \partial^k B^k = 0$. If $\mathbf{m}$ is constant, then it can be brought inside the partial derivative, and we finally obtain the expression for the force on a magnetic dipole as given in Griffiths,

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}). \quad (47)$$

Note however that if $\mathbf{m}$ is not constant, for instance if it is an induced dipole moment, then we cannot pass from (43) to (47).