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Runaway solutions and pre-acceleration

With the radiation reaction force $\mathbf{F}_{rad} = (\mu_0 q^2/6\pi c)\dot{\mathbf{a}}$, the equation of one dimensional motion in the presence of an external force $\mathbf{F}(t)$ takes the form

$$a - \tau\dot{a} = f(t), \tag{1}$$

where $\tau = \mu_0 q^2/6\pi cm$, and $f(t) = F(t)/m$, where m is the mass of the particle. This is equation is third order in time derivatives of position, so a solution is determined by initial position, velocity and acceleration. In terms of $a(t)$ however it is first order, and can be solved with one undetermined constant. Given that solution, the position can be found given an initial position and velocity. Note that the equation does *not* have time reversal symmetry.

To solve (1), let $a(t) = e^{t/\tau}a_1(t)$, so $a - \tau\dot{a} = -\tau e^{t/\tau}\dot{a}_1$. In terms of a_1, (1) takes the form

$$\dot{a}_1 = -\tau^{-1}e^{-t/\tau}f(t), \tag{2}$$

which has the general solution

$$a_1(t) = a_1(\infty) + \tau^{-1}\int_t^\infty e^{-t'/\tau}f(t')\,dt'. \tag{3}$$

I've written the solution so that the value of a_1 at $t = \infty$ is the constant of integration. The actual acceleration is then

$$a(t) = e^{t/\tau}a_1(\infty) + \tau^{-1}\int_t^\infty e^{(t-t')/\tau}f(t')\,dt'. \tag{4}$$

Looking at this solution, the acceleration blows up as $t \to \infty$ unless we choose the solution that has $a_1(\infty) = 0$. This final state boundary condition excludes the so-called *runaway solutions*. But then notice that the acceleration at time t depends on the force at times $t' > t$. This is referred to as *pre-acceleration*. The dependence is "controlled" however in the sense that the contribution to $a(t)$ from the force at time $t' > t$ is suppressed by the factor $e^{(t-t')/\tau}$.

As a concrete example, consider the case where the force is a delta function impulse, $f(t) = V\delta(t)$. Then the solution (4) becomes

$$a(t) = e^{t/\tau}a_1(\infty) + \tau^{-1}Ve^{t/\tau}\,\theta(-t) \tag{5}$$

The only solution that doesn't blow up is the one with $a_1(\infty) = 0$,

$$a(t) = \tau^{-1}Ve^{t/\tau}\,\theta(-t), \tag{6}$$

which grows exponentially up to V/τ at $t = 0$, after which it drops to zero.

What is this time τ? We may re-write its defining equation as $\tau = (4/3)r_q/c$, where $r_q = (q^2/8\pi\epsilon_0)/mc^2$ is the "classical radius" of the charge q, i.e. the radius outside of which the electric field energy is equal to the mass energy. So τ is basically the time it would take light to travel the classical radius. For an electron one has $\tau = 6 \times 10^{-24}$ s.