HW#5 —Phys374—Spring 2007 Prof. Ted Jacobson
Due before class, Friday, March 2, 2007 Room 4115, (301)405-6020
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1. Derive the identity
$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v} \tag{1}$$
where f is a scalar field and $\mathbf{v}$ is a vector field. [5 pts.]

2. It was claimed in class Friday that if f is a function of one variable, and if h is a function of position $\mathbf{r}$, then $\nabla f(h) = f'(h)\nabla h$. Show explicitly that this is true, by showing that the vector components of each side of the equation are equal. [5 pts.]

3. Evaluate the expression
$$\nabla \times (f(r)\mathbf{r}), \tag{2}$$
where $\mathbf{r}$ is the position vector from the origin to the point $\mathbf{r}$, and $r = |\mathbf{r}|$, using (i) Cartesian coordinates and (ii) spherical coordinates (cf. (7.16)). [3+2=5 pts.]

4. Problem 7.2 (d) only. Do this using (i) Cartesian coordinates, as indicated in the problem, and (ii) cylindrical coordinates, using (7.17). (Note that in this and the next two problems r is the cylindrical radius.) [3+2=5 pts.]

5. Problem 7.3 (a),(b) only. [3+2=5 pts.]

6. Problem 7.4 (d) only, plus (e) and (f) below. (The flow is (7.6), now with $v(r) = A/r$.)
(e) Use Stokes' theorem and the result ($\nabla \times \mathbf{v} = 0$) of part (d) to show that $\oint_{C_r} \mathbf{v} \cdot d\mathbf{l}$ is independent of r, for any circle C_r at fixed r and z. (*Hint: Apply Stokes' theorem to the annular region*[1] *between two radii* r_1 *and* r_2 *at constant* z.)
(f) Compute $\oint_{C_r} \mathbf{v} \cdot d\mathbf{l}$ directly for the given form of $\mathbf{v}$ and show that it is indeed independent of r as shown in (e), and is non-zero.
[3+5+2=10 pts.]

7. Consider the flow in a pipe extended along the z direction, with flow velocity $\mathbf{v} = v(\rho)\hat{\mathbf{z}}$, where $r = (x^2 + y^2)^{1/2}$ is the cylindrical radius coordinate ($x = y = 0$ being the central axis of the pipe). (a) Compute the divergence and curl of $\mathbf{v}$ for this flow, using both (i) Cartesian, and (ii) cylindrical coordinates. (b) What is the physical meaning of your result for the divergence? (c) What direction does the curl point in at each location? (*Hint*: The direction depends on position.) [3+3+2+2=10 pts.]

8. Problems 9.6 a,b,c (*Wingtip vortices*) [1+1+3=5 pts.]

9. In the supplement on Maxwell's equations, equation (14) expresses conservation of electromagnetic field energy plus charged particle energy. Derive this equation from Maxwell's equations. (*Hint*: The supplement gives some guidance.) [10 pts.]

[1] If you apply Stokes' theorem to a disk of radius r you seem to conclude that $\oint_{C_r} \mathbf{v} \cdot d\mathbf{l} = 0$, but as you see in part (f) this is not the case. The resolution of this puzzle is similar to what we saw in class Friday for the application of Gauss's theorem to the gravitational field of a point mass: here the flow velocity field $\mathbf{v}$ is singular at $r = 0$, and there is a singular contribution to $\nabla \times \mathbf{v} = 0$ at the origin. Since this lies outside the annular region considered in part (e) it does not affect that argument.