HW#13 —Phys374—Spring 2007 Prof. Ted Jacobson
Due before class, Wednesday, May 9, 2007 Room 4115, (301)405-6020
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1. Consider a mass m suspended by a spring with spring constant k from another mass m which is suspended from a fixed support by another spring with spring constant k. Let y_1 and y_2 denote the displacements from their equilibrium positions of the top and bottom masses respectively, with the downward direction taken as positive.

   (a) Write out Newton's second law governing the displacement of each of the two masses from their equilibrium positions.

   (b) Combine the two Newton's law equations into a single 2×2 matrix equation $\ddot{\mathbf{y}} = A\mathbf{y}$, and specify the components of the matrix A in units with $k = m = 1$.

   (c) Determine the normal mode amplitudes and frequencies by finding the eigenvectors and eigenvalues of A *by hand*. (*Hint*: To check yourself, the squared frequencies are $(3 \pm \sqrt{5})/2$.)

   (d) Describe or indicate with arrows the nature of the two normal mode motions, showing both direction and relative amplitude of the motion of each mass. Indicate which normal mode has the higher frequency. Using the concept of effective spring constant described below, explain the ordering of the frequencies and explain qualitatively how the effective spring constants of the two masses are equal in each of the two normal modes.

2. Re-do all parts of the previous problem in the case that there are three equal masses hanging in a chain from three springs. In this case the characteristic equation is a cubic and the frequencies and amplitudes cannot be found in a simple closed form, so instead use a computer to get them.[1] For part (d), discuss all three modes, and explain, using the equality of effective spring constants, why for the middle frequency mode the top two masses move together while bottom mass moves opposite, rather than the bottom two together with the top opposite.

3. Find the ratio of the two nonzero normal mode frequencies for linear vibrations of the model of the carbon dioxide molecule discussed in class: two oxygen mass points with mass m_O each connected by a spring of spring constant k to a carbon atom in the center with mass $m_C = (3/4)m_O$. (The equilibrium length of the springs is nonzero.) We showed in class that if all masses were equal the frequencies would be $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$, so $\omega_2/\omega_1 = \sqrt{3}$. The point of this problem is to find the ratio taking into account that $m_O/m_C = 4/3 \neq 1$. Use units with $k = 1$ and $m_O = 1$, and write your equations using the parameter $a = m_O/m_C$. Put $a = 4/3$ only at the end of your calculation to evaluate the result.

   (a) Use the characteristic polynomial to show *by hand* (no computer or calculator) that the frequencies are given by 0, 1, and $\sqrt{1 + 2a}$.

[1] In Mathematica, the command would be, for example, "Eigensystem[{{1., 2, 3},{0, 1, 0},{1, 0, 1}}]". Note the period after the first entry 1. That tells Mathematica to treat the entries as real numbers rather than integers, to make it give a numerically computed result rather than an abstract "root" expression.

(b) Look at www.phy.davidson.edu/StuHome/jimn/CO2/Pages/CO2Theory.htm for information on carbon dioxide vibrations, and compare the ratio $\sqrt{11/3}$ of the frequencies in the model to the measured value for the corresponding modes of carbon dioxide. Which is larger, and by what fraction do they differ? (Note you don't need to know the spring constant or the magnitude of each mass to do this! Also note that frequency is given there in units of cm^{-1}, which refers to the frequency of the radiation that is in resonance. What is indicated is the inverse wavelength λ^{-1} of the radiation, which is proportional to the frequency ω since $\omega = ck = 2\pi c/\lambda$, where c is the speed of light.)

(c) Try to give a physical reason (or reasons) for the difference between the prediction of our simple model and the observed ratio. One reason might be "quantum mechanics versus classical mechanics", which I think is not really the important issue in this case, so ignore this reason. *Hint*: Think about the physical nature of the "springs" in the molecule and how they differ from those in our model.

Effective spring constant and normal modes

In a normal mode, the frequency of motion is the same for all the masses and the displacements of all the masses have fixed ratios. The displacement one mass therefore determines that of all the other masses in the system, so the force on the ith mass with displacement x_i may be written as $-k_{\mathrm{eff},i}\, x_i$. The "effective spring constant" $k_{\mathrm{eff},i}$ is determined by the combined effect of the stretch or compression of the springs connected to the ith mass in a given normal mode, and the frequencies $\sqrt{k_{\mathrm{eff},i}/m_i}$ are all equal. In particular, if the masses are equal then the effective spring constants must also be equal. This can help in identifying the normal mode motions and determining the ordering of their frequencies.