HW#12 —Phys374—Spring 2007
Due before class, Friday, May 4, 2007
www.physics.umd.edu/grt/taj/374b/

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1. **Sampling Theorem**

> Exact reconstruction of a continuous-time signal from its discrete-time samples is possible if the signal is band-limited and the sampling frequency is greater than twice the signal bandwidth.

Consider a signal $f(t)$ whose Fourier transform $\tilde{f}(\omega)$ is zero for $|\omega| > \Omega$,

$$f(t) = \int_{-\Omega}^{\Omega} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega. \tag{1}$$

This is called a *band-limited* signal. Evaluating (1) at the discrete times $t = nt_s$, where the *sampling time* t_s is defined by $t_s = \pi/\Omega$, yields

$$f(nt_s) = \int_{-\Omega}^{\Omega} \tilde{f}(\omega)\, e^{-in\pi\omega/\Omega}\, d\omega. \tag{2}$$

The right hand side of (2) is recognized as 2Ω times the nth coefficient in the Fourier series for $\tilde{f}(\omega)$. Being limited to the finite range $-\Omega < \omega < \Omega$, the function $\tilde{f}(\omega)$ is determined by its Fourier series coefficients, and therefore by the discrete "samples" $f(nt_s)$. The sample values thus determine $f(t)$ via (1). The sampling frequency $1/t_s = \Omega/\pi$ is twice the bandwidth $\Omega/2\pi$.

Show that $f(t)$ can be reconstructed explicitly from the samples $f(nt_s)$ via

$$f(t) = \sum_{n=-\infty}^{\infty} f(nt_s)\, \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{3}$$

2. **Electron tunneling between neighboring layers: an application of Dirac delta functions**

When electrons tunnel (make transitions) between two closely spaced semiconducting layers in the presence of a magnetic field in the y direction, the transition probability is proportional to the integral

$$D = \iint \delta\left(\frac{p_x^2 + p_y^2 - p_F^2}{2m}\right)\, \delta\left(\frac{(p_x - q)^2 + p_y^2 - p_F^2}{2m}\right)\, dp_x\, dp_y, \tag{4}$$

where the δ's are Dirac delta functions. (See Chapter 14. Section 14.3 discusses delta function of a function, which I explained in class Friday.) p_F is the *Fermi momentum*, m is the electron mass, and $q = eBd$ is the product of electric charge e, magnetic field B, and inter-layer distance d. The factor D determines interlayer conductivity, which can be measured experimentally as a function of the magnetic field.

(a) Make a sketch in the (p_x, p_y) plane to show geometrically where the two delta functions are non-zero for different values of q.

(b) Show using dimensional analysis that D has dimensions of $[m/p_F]^2$. (This will require figuring out what are the dimensions of the delta functions. *Hint*: Think about the normalization condition defining the delta function.)

(c) Calculate D. For practice, and to simplify writing, adopt units with $p_F = 1$ and $m = 1$, and restore the factors of p_F and m at the end using dimensional analysis. (*Hint*: First do one integral, then the other.)

(d) Sketch a plot of D as a function the dimensionless parameter q/p_F, and explain the qualitative properties of the dependence on q in terms of your sketch in part 2a. (*Guidance*: Imagine the delta functions as having a small width, before taking the limit, and consider how the area of the region in which both delta functions are non-zero depends on q.)

3. **Wavepackets and group velocity for a relativistic quantum particle**

In relativistic quantum mechanics, the (complex) wave function Φ for a particle of mass m satisfies the partial differential equation

$$\partial_t^2 \Phi = c^2 \partial_x^2 \Phi - (m^2 c^4/\hbar^2)\Phi, \tag{5}$$

where c is the speed of light and $\hbar$ is Planck's constant. This is called the *Klein-Gordon* equation. For simplicity it is assumed here that the wave function depends on only one space coordinate x.

(a) Assume $\Phi(x, t)$ has the form of a complex exponential $e^{-i\omega t} e^{ikx}$ with definite frequency and wave vector, and determine the dispersion relation $\omega(k)$ between the frequency and wave vector by imposing the equation (5).

(b) Using the quantum mechanical relation $E = \hbar\omega$ between energy and frequency and $p = \hbar k$ between momentum and wavevector, rewrite the dispersion relation as a relation between energy and momentum. Planck's constant should not appear. (See problem 4 of Exam 1 if you want to check yourself.)

(c) Write an expression for a completely general positive frequency ($\omega > 0$) wavepacket solution to (5). What free function determines your general solution?

(d) What is the group velocity of a purely positive frequency wavepacket composed of a narrow range of wavevectors centered on some value k_0? Does it depend on k_0?

(e) Now suppose the particle is massless, $m = 0$. What is then the group velocity of a purely positive frequency wavepacket composed of a narrow range of wavevectors centered on some value k_0? Does it depend on k_0?