HW#2 —Phys374—Spring 2006 Prof. Ted Jacobson
Due before class, Monday, Feb. 13, 2006 Room 4115, (301)405-6020
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1. Consider the quadratic equation $ay^2 + y + 1 = 0$.
   (a) Display the location of the roots graphically, by sketching the line $y + 1$ and the curve $-ay^2$ and seeing where they intersect. Show in your sketch three cases, where a is small compared to 1, intermediate, and large compared to 1. Discuss qualitatively the existence and location of the roots in these three cases, based only on your sketch. [3 pts.]
   (b) Find the roots $y_\pm$ using the quadratic formula. [1 pt.]
   (c) Determine the leading order a dependence of the roots in the limit $a \ll 1$. [3 pts.]

2. **Relativistic Energy**: The total energy of a free particle of mass m and speed v is given in special relativity by
$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{1}$$
where c is the speed of light. (i) Re-write this relation using units with $m = c = 1$. (ii) Write out the Taylor expansion of the resulting expression, using $x = v^2$ (not v) as the variable. Write explicitly the terms out to $O(v^6)$. (iii) Re-write the answer to (ii) in general units by putting back in the appropriate factors of m and c as determined by dimensional analysis. (iv) Identify the physical meaning of the first two terms in the series. (v) For an electron in the ground state of a hydrogen atom $v \sim 0.01c$. Based on your expansion, what order of magnitude would you expect the relativistic corrections to the ground state energy to be compared to the non-relativistic value? [1+2+2+2+2=9 pts.]

3. **Speed of surface waves on shallow or deep water**

   In HW#1 you used dimensional analysis to find the speed of surface waves on water. When surface tension is irrelevant, and gravity is the only restoring force, these are called "gravity waves". The result was $v \propto \sqrt{gh}$ for shallow water ($h \ll \lambda$) and $v \propto \sqrt{g\lambda}$ for deep water ($\lambda \ll h$). It turns out that the formula interpolating between these two is
$$v_{ph} = \sqrt{\frac{g \tanh kh}{k}} \tag{2}$$
where $k = 2\pi/\lambda$ is the wave vector. (i) In the intermediate regime, in addition to the dimensionful quantities there is a dimensionless number that characterizes the wave speed. What is that dimensionless number? (ii) Show explicitly that (2) reduces in the shallow and deep limits to the forms quoted above. [2+2=4 pts.]

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4. **Lagrange points L1 and L2**

If the earth's orbit around the sun were perfectly circular, there would be a point L1 inside the earth's orbit where a satellite (of negligible mass) would orbit the sun with precisely the same angular velocity as the earth, since the extra pull of the sun would be canceled by the pull of the earth (the gravitational pull of all other bodies in the solar system being neglected). There would be a similar point L2 just outside the earth's orbit where the smaller pull of the sun would be supplemented by the pull of the earth. These are called "Lagrange points".

(a) Write an equation expressing the condition that the orbital frequency at L1 is equal to that of the earth, and similarly one for L2. Your equations should be based on Newton's law of gravity and might involve Newton's constant G, the masses of the sun and the earth m_s and m_e, the distance d_s from the sun to the earth, and the distance $d_{1,2}$ from the earth to the Lagrange points. (*Hint*: recall that centripetal acceleration on a circle can be written as $r\omega^2$.) [3 pts.]

(b) Re-write your answer to the previous part using only the dimensionless ratios $\mu = m_e/m_s$ and $\delta_{1,2} = d_{1,2}/d_s$. This should lead you to the equation we considered in class Wednesday, Feb. 2 for the case of L1. [1 pt.]

(c) Find the solutions for $\delta_{1,2}$ to leading order in μ. (They should be identical.) You most likely already found this in class, but for this homework write out explicitly your method of solution. One approach is to first express the equation as a fifth order polynomial equation, and then ascertain which terms give the leading order solution. [3 pts.]

(d) Put in the mass ratio $\mu = 1/332,830$ and find $\delta_{1,2}$, then put in the (mean) solar distance $d_s \simeq 1.5 \times 10^8$ km and find $d_{1,2}$ in km. [2 pts.]

(e) Show that the *next to leading* order term in $\delta(\mu)$ is $-(1/3)(\mu/3)^{2/3}$. You can do this by writing $\delta = \delta_0 + \epsilon$ where δ_0 is the leading order term you found in part 4c and ϵ is the next to leading order term, and once again ascertaining which terms give the leading order solution for ϵ. How large is this correction compared to the leading order term? [3 pts.]