Bounding Curvature by Modifying General Relativity

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The existence of singularities is one the most surprising predictions of Einstein's general relativity. I review some work that tries to eliminate singularities in the theory by forcibly bounding curvature scalars. Although the authors of the original papers sought a possible effective theory in high curvature regions that might help develop either quantum gravity or string theory, the technical difficulties of calculating field equations within the modified theories is quite formidable and likely makes it impossible to gain any analytic insights. I also briefly discuss possible applications of these methods to numerical relativity.

I. INTRODUCTION

General relativity admits solutions that include essential, physical singularities. Although these singularities likely indicate a break down of the classical theory, the "next" theory has not yet been fully developed. Given the lack of a non-perturbative theory describing regions of high curvature, some authors have been motivated to examine corrections to general relativity that bound certain curvature invariants to values less than appropriate powers of the Planck length, both in the context of cosmologies [1, 2] and in the context of low-dimensional black holes [3]. This work was motivated by the hope that some features of the more fundamental theory can be found by such considerations and by the hope that predictions of this ad hoc theory might help in the formulation of the fundamental theory.

The problem of singular solutions is also of interest from a less fundamental point of view. In the context of numerical relativity, where one would like, for example, to compute the gravitational wave forms propagating away from black hole binary systems in support of experiments such as LIGO and LISA, singularities present a difficult technical challenge. It is not possible to represent a singularity on the computer, and, even in the neighborhood of a singularity, steep gradients are not handled well by finite-difference approximations, which can lead to code crashes even if one manages to "hide" the singularity itself. In practice, this problem has been handled either by using gauge conditions to avoid the singularities, removing a large region inside of the black hole event horizon (see, e.g., [4, 5]), or by taking "puncture" initial data [6]. Although some of these methods have been developed quite successfully, modifying the Einstein equations in a way that leaves the theory invariant in the low curvature regime while eliminating the singularities at the centers of black holes, could be an interesting alternative to the current computational tools. In this context, the most like candidate for a length scale is the discretization length rather than the Planck length.

Before proceeding further, it is worth noting exactly which type of singularities that I will discuss here. Wald, for example, identifies three classes of singularities [7]:

1. Scalar curvature singularities at which some scalar formed from the Riemann tensor and its covariant derivatives blows up at the point in question,
2. Parallelly propagated curvature singularities at which the first condition does not hold, but a component of Riemann or its covariant derivatives blows up as measured relative to a parallelly propagated tetrad, and
3. Any other point in the manifold which is an end point of a geodesic.

Because the methods that I will describe force curvature scalars to obey some specified bounds, I will only be able to address singularities of the first type. This, however, is sufficient for many cosmological cases of interest since many of the known analytic solutions to the Einstein equations have singularities of the first type. It is also completely sufficient for any potential application to numerical relativity that might be considered in the near future.

Even within this class of scalar curvature singularities, there still remains a problem in explicitly bounding individual curvature scalars, namely that bounding low order invariants does not guarantee that higher order invariants will be bounded. In principle this leaves an infinite number of scalars to bound, which would be a difficult or impossible task. This technical problem can be avoided by introducing the "limiting curvature hypothesis" (LCH) (see e.g. [8] and references therein), which states that a finite number of curvature invariants are bounded, and, when those scalars achieve their bounds, the solutions to the field equations become a specified, non-singular solution. The work of [1, 2, 3], on which I will focus here, takes the specified spacetime to be de Sitter.

II. FORMALISM

In order to proceed, I need to introduce the formalism in which the curvature bounds will be specified. I begin by looking at the problem of specifying bounds on dynamical variables within a Lagrangian formulation, and then look at the problem of the specific bounds needed for general relativity.

A. Inserting bounds into a Lagrangian

In order to successfully bound the curvature invariants, I need to develop a mathematical tool. Bounds may be mathematically enforced in a system by introducing non-dynamic scalar fields into the Lagrangian of the theory [9]. Let $\mathcal{L}_0(\psi)$ represent the Lagrangian density of an unbounded theory on a field ψ, and let f be some function of ψ that must satisfy some specified bound $\sup f = B$. Then taking the new Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \phi f(\psi) - V(\phi) \tag{1}$$

will force f to obey the specified bound provided that $V(\phi)$ is chosen such that

$$\sup \frac{\partial V}{\partial \phi} = B. \tag{2}$$

To see that this is true, consider the variation of the associated action with respect to the new field ϕ. It yields the constraint

$$f(\psi) = \frac{\partial V}{\partial \phi} \tag{3}$$

so that if the partial derivative on the right hand side is bounded, so is f.

As a concrete case, following [1, 2], consider the action

$$S = m \int \left[\frac{1}{2}\dot{x}^2 + \phi \dot{x}^2 - V(\phi) \right] dt \tag{4}$$

with

$$V(\phi) = \frac{2\phi^2}{1+2\phi} \tag{5}$$

giving the potential. The variation with respect to ϕ yields the constraint

$$\dot{x}^2 = 1 - \frac{1}{(1+2\phi)^2} \tag{6}$$

from which is it clear that $\dot{x}^2 \leq 1$. In fact, solving (6) for ϕ and substituting it back into the action (4) shows that, with the chosen potential, (4) differs from the standard action for a particle in special relativity only by a constant.

B. Bounding curvature invariants

I now turn to applying the methods of the previous subsection to the Einstein equations. In order to do so, I need to choose N curvature invariants $\{I_n\}_{n=1}^N$ that should satisfy bounds $f_n = f_n(I_n) \leq B_n$ for some specified functions f_n and specified constants B_n. I then take a modified Hilbert action

$$S = -\frac{1}{16\pi} \int \left[R + \phi_n f_n - V(\phi) \right] \sqrt{-g}\; d^4x \tag{7}$$

with Lagrange multiplier fields $\{\phi_n\}_{n=1}^N$ and a potential, which is unspecified at this point in the derivation, $V(\phi) = V(\phi_1, \ldots, \phi_N)$ [12]. For a given problem, one must proceed through the following steps:

1. Identify appropriate curvature scalars to bound,

2. Choose a potential that enforces those bounds and gives back standard general relativity in the low curvature limit,
3. Determine the "dynamics" of the non-dynamical Lagrange multiplier fields $\{\phi_n\}_{n=1}^N$, by solving the constraints

$$\frac{\partial V(\phi)}{\partial \phi_n} = f_n(I_n) \tag{8}$$

(which come from the variation of the action (7) with respect to ϕ_n) for the Lagrange multipliers in terms of the true dynamical variables in the theory, and

4. Evolve the fundamental degrees of freedom forward in time using the field equations derived from the modified action.

In the context of cosmology, Brandenberger, Mukhanov, and Sornberger [2] give sufficient conditions that allow one to accomplish the first two steps. They leave even the possibility of successfully completing the third step in all but some very special cases as an open question. In a later paper, Trodden, Mukhanov, and Brandenberger [3] demonstrate an explicit construction of this type of theory for black holes in $1+1$ spacetime. I look at each of these constructions in the next two subsections.

C. Application to cosmology

In order to apply the LCH to an homogeneous, isotropic universe, two curvature invariants must be bounded. One may be bound in such away as to limit curvature, while the second bound can be chosen to force the system to a fixed solution when the bound on the first invariant is saturated. Following [2], I choose

$$I_1 = R - \sqrt{3(4R_{ab}R^{ab} - R^2)} \tag{9}$$

and

$$I_2 = 4R_{ab}R^{ab} - R^2 \tag{10}$$

where R is the Ricci scalar. The form of I_1 is convenient for cosmology because it is simply related to H as $I_1 = 12H^2$. If the potential $V(\phi)$ could be chosen such that $I_2 \to 0$ as $|\phi_2| \to \infty$, assuming as I do in all of this subsection that large ϕ_i corresponds to regions of high curvature in Einstein gravity, then the form of I_2 would pick out de Sitter space as the "high curvature limit" of the theory since, for homogeneous and isotropic spacetimes, $I_2 = 0$ only for de Sitter space.

In order to further simplify the situation, assume that the potential factorizes

$$V(\phi_1, \phi_2) = V_1(\phi_1) + V_2(\phi_2). \tag{11}$$

Under this assumption, analysis of the asymptotic behavior of the two pieces of the potential can be more easily analyzed. In order to recover the Einstein equations in the low curvature limit, the leading order behavior of both fields must satisfy

$$V_i(\phi_i) \sim \phi_i^2, \;\; |\phi_i| \ll 1 \tag{12}$$

since this will take both the interaction terms and the potential terms in the modified action to zero quadratically for small ϕ_i. In order to limit R, Brandenberger et al. suggest forcing

$$V_1(\phi_1) \sim \phi_1, \;\; |\phi_1| \gg 1 \tag{13}$$

since this bounds $f_1(I_1) \to 1$ as $|\phi_1| \to \infty$. In order to ensure that solutions go to de Sitter solutions at this limit, we must bound I_2 at 0. This means assuming that $f_2(I_2) \to 0$ as $I_2 \downarrow 0$, and choosing the potential V_2 such that

$$V_2(\phi_2) \sim \text{constant}, \;\; |\phi_2| \gg 1. \tag{14}$$

Even under these assumptions, Brandenberger et al. leave open the question of whether such a potential can be chosen in a way that guarantees the possibility of solving those equations for the fields ϕ_i. As an analytic problem, this is quite formidable since the constraints (8) will generically depend in a very non-linear way on the dynamical fields of the theory. One might hope to complete the task numerically either by solving (8) directly, or by adding "driver" terms to the evolution equations derived from the action principle that asymptotically enforce the constraint given by (8) [10].

D. Application to black holes in $1+1$ dimensions

In the case of a static $1+1$ dimensional spacetime, a completely explicit calculation of a non-singular theory can be computed along the lines described here. This result, however, is somewhat unsatisfying since the authors employed tricks that do not seem to generalize to higher dimensional problems. I will summarize the results here, without delving into the details of these specific tricks.

To begin, they consider a Lagrangian of the form

$$\mathcal{L} = \sqrt{-g}\,[V(\phi) + D(\phi)R] \tag{15}$$

(which they attribute to [11]) for an interacting graviton-dilaton field theory. While Brandenberger et al.'s exact reason for choosing this form of for the Lagrangian is not clear, note that it can easily be transformed to the form of (1) by taking $D(\phi) = 1 + \phi$, where the Ricci tensor is at once generating the Einstein gravity portion of the theory, and the curvature invariant that is bounded (c.f. (4) where $\dot{x}^2$ played a similar dual role for special relativity). In this context, ϕ is a Lagrange multiplier in the sense of the previous sections, and *not* a dilaton in the sense of usual dilaton gravity. From this point, the authors then use the fact that they are working in two dimensions so that, invoking gauge freedom, they can assume their metric to have diagonal form with $g_{tt} = -f(r)$, and $g_{xx} = 1/f(r)$.

Furthermore, by redefining $\phi \to 1/(1+\phi)$, the authors take $D(\phi) = 1/\phi$. In principle this also implies changing the functional form of V, but that has not yet been specified anyway. This redefinition subtly mixes the roles of the different terms in the action. Since R no longer appears by itself in any term, this means, in particular, that the potential V must be chosen such that terms generated by variations of it with respect to ϕ contribute to the field equations in the low curvature limit in just the right way to get the Einstein equations back in that limit. This is in contrast to the cosmological calculation in which the corrections to the original Hilbert action were simple additions of correction terms, and those correction terms where chosen so that they turned off in the low curvature limit. Nonetheless, the "mixed" approach followed here is essentially equivalent to the approach followed in previous sections, even if conceptually less clear.

The assumptions made possible due to the low dimensionality of the spacetime with the form of D chosen above reduce the number of field equations to two, subject to the constraint

$$\frac{\partial V}{\partial \phi} = \frac{R}{\phi^2}. \tag{16}$$

The simple form of the metric, combined with the fact that they only have *ordinary* differential equations, allows them to manipulate both the constraint and the field equations to write a differential equation for the scalar field

$$\frac{\phi}{\phi'}\left(\frac{\phi'}{\phi}\right)' - \frac{\phi'}{\phi} = 0. \tag{17}$$

In sufficient generality, this has the solution $\phi = 1/Ar$ for arbitrary A. At this point, they have constructed the field ϕ as a function of the coordinates (it does not depend on time here because they are only considering static solutions), so they have overcome the seemingly insurmountable obstacle encountered in the cosmological setting. Note, however, how many assumptions had to be inserted by hand in order to reach this point.

Having found an explicit expression for ϕ, the path to a solution is now clear. The potential $V(\phi)$ must be chosen so that for large r (small ϕ), the solution is Schwarzschild, while for small r (large ϕ), the solution has bounded curvature. Once this is accomplished, the exact form of V can be used in the field equations for f to solve for the metric exactly.

In order to proceed further, I need first to rewrite the constraint equation (16) in terms of the function f. In $1+1$ theory, the Ricci scalar is simple, so that

$$\frac{\partial V}{\partial \phi} = -\frac{f''(r)}{\phi^2}. \tag{18}$$

Because of (18) and the functional form of ϕ, demanding that $f(r) = 1 - 2m/r$ for large r (small ϕ), means demanding that

$$V(\phi) \sim 2mA^3\phi^2(r), \;\; r \gg 2m \tag{19}$$

in that region. In performing the integration required to go from (18) to (19), it is safe to ignore the constant of integration since a constant addition to the potential does not affect the physics. In the other limit, one demands that R remain bounded. From (16) it is clear that taking $\partial V/\partial\phi \sim \phi^{-2}$ in that limit will bound $R \leq 1$. Integrating that relationship says

$$V(\phi) \sim \frac{2}{l^2\phi}, \quad r \ll 2m \tag{20}$$

in terms of another integration constant l. One possible way to interpolate between these limiting values is to take

$$V(\phi) = \frac{2mr}{1 + mA^3 l^2 \phi^3} \tag{21}$$

for the potential. Using this value gives a differential equation for f

$$f'(r) = \frac{2mr}{r^3 + ml^2} \tag{22}$$

which is integrable between r_0 and r to give

$$\begin{aligned} f(r) \;=\; & \frac{1}{3}\left(\frac{m}{l}\right)^{2/3} \log\left[\frac{r^2 - (ml^2)^{1/3} r + (ml^2)^{2/3}}{r_0^2 - (ml^2)^{1/3} r_0 + (ml^2)^{2/3}} \left(\frac{r_0 + (ml^2)^{1/3}}{r + (ml^2)^{1/3}}\right)^2\right] \\ & + \frac{2}{\sqrt{3}}\left(\frac{m}{l}\right)^{2/3}\left[\arctan\left(\frac{2r - (ml^2)^{1/3}}{\sqrt{3}(ml^2)^{1/3}}\right) - \arctan\left(\frac{2r_0 - (ml^2)^{1/3}}{\sqrt{3}(ml^2)^{1/3}}\right)\right] \end{aligned} \tag{23}$$

for the final form of the metric.

III. CONCLUSIONS AND COMMENTS

The methods described here allow, in principle at least, one to write non-singular solutions of a theory which is Einstein gravity in the low curvature limit and something new in the "high curvature limit." These solutions are constructed by following the least curvature hypothesis, and have the feature that regions of spacetime that were singular in the Einstein theory are replaced by segments of de Sitter spacetime in the modified theory. The authors who originally presented the LCH and the methods described here argued that such a process might reveal qualitative features of the fundamental theory that takes over in high curvature regions, though it seems the technical difficulties in constructing solutions to the modified equations is too formidable in all but the most simple cases to yield any useful insight.

As a numerical tool, such methods might be useful, but their usefulness would depend on the ability to construct a potential that is invertible in the sense that the constraints on the auxiliary fields can be used to determine the values of the auxiliary fields in terms of the fundamental fields of the theory. Because the field equations themselves for the modified theory may be *much* more complicated than for Einstein gravity, however, such an approach would need to be compared to existing methods for handling the singularities to see if the extra time in terms of coding and in terms of performance would justify this procedure. Changes to the field equations that introduce more possible solutions, in addition, could exacerbate other problems frequently encountered in numerical studies of the Einstein equations.

Aside from these practical concerns, it is interesting, as a final note, to point out that the cosmological constant of the de Sitter solution that takes the place of the high curvature regions in the Einstein theory is not, in these cases, put in arbitrarily, but arises out of the dynamics of the modified theory.

[1] A. Mukhanov and R. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992).
[2] R. Brandenberger, V. Mukhanov, and A. Sornborger, Phys. Rev. **D48**, 1629 (1993).
[3] M. Trodden, V. F. Mukhanov, and R. H. Brandenberger, Phys. Lett. **B316**, 483 (1993).
[4] M. Alcubierre and B. Brügmann, Phys. Rev. **D63**, 104006 (2001).
[5] D. Shoemaker, K. Smith, U. Sperhake, P. Laguna, E. Schnetter, and D. Fiske (2003), submitted to *Class. and Quant. Grav.*, gr-qc/0301111.
[6] S. Brandt and B. Brügmann, Phys. Rev. Lett. **78**, 3606 (1997).
[7] R. M. Wald, *General Relativity* (University of Chicago Press, 1984).
[8] V. P. Frolov, M. A. Markov, and V. F. Mukhanov, Phys. Rev. **D41**, 383 (1990).
[9] B. L. Altshuler, Class. and Quant. Grav. **7** (1990).

[10] D. R. Fiske (2003), submitted to *Phys. Rev. D*, gr-qc/0304024.
[11] T. Banks and M. O'Loughlin, Nucl. Phys. **B362**, 649 (1991).
[12] I have recast the analysis of [2] into terms that I consider more succinct. Note in particular that [2] appears to have a sign inconsistency between the last term of its equation 2.9 and equations 2.10–2.11. I have, I hope, avoided that mistake here.