Black Holes and the Weak Field Limit
In General Relativity with a Preferred Frame

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Over the past several years the increasing recognition of the role of the Planck scale as a fundamental cutoff length as lead a number of researchers to seriously consider violations of Lorentz invariance. Generally, such violations are considered within the realm of Quantum Field Theory, but it is also interesting to examine the effects of Lorentz violating terms on General Relativity (GR). One way to do this would be to postulate some fixed background structure as the "aether field" or preferred frame of the universe. However, such structures by definition break the general covariance of GR, a feature many physicists feel is essential for a consistent theory. Thus, one is naturally led to a theory of a dynamical aether, one that is affected by the behavior of all the spacetime fields. In this report I consider the weak and strong field consequences of the dynamical aether model of Jacobson and Mattingly [1,2] with the total Lagrangian given by the aether piece

$$L_{ether} = [c1\, g^{ab} g_{mn} + c2\, \delta^a{}_m \delta^b{}_n + c3\, \delta^a{}_n \delta^b{}_m + c4\, g_{mn} u^a u^b]\, u^m{}_{;a}\, u^n{}_{;b} \qquad (1)$$

plus the usual Ricci scalar, where the u vector field defining the preferred frame is constrained to be unit timelike. Note that the coefficients c1-c4 are dimensionless parameters giving the "weight" of each term in the action and should be fairly small in order to agree with experiment. This model is similar to the Will-Nordtvedt-Hellings vector-tensor theory described in [3], but the presence of the constraint and the c4 term should introduce novel effects. Specifically, can the coefficients be bounded by basic solar system observations and can the theory be ruled by the fact that it does not allow for the existence of black holes? Using the tensor package of Maple I analyzed the asymptotic limit of the spherically symmetric solution and tested for local solutions to the field equations in the neighborhood of an event horizon. With the asymptotic limit I was able to directly find the two Eddington-Robertson-Schiff parameters and obtain an initial bound on the coefficients of the theory given in the above Lagrangian. The analysis of local solutions near an event horizon seems to show that black holes are permitted for the c1-c3 terms but not allowed for the c4.

Asymptotic Weak Field Limit

Field Equations

The starting points for the analysis of the aether theory are the field equations, which can be obtained from the action by the usual variations with respect to the metric and the u field. In particular, the metric equations can be written in the form

$$G_{ij} = T_{ij\_ether} \tag{2}$$

Where T_{ij_ether} is the variation of the u part of the Lagrangian given above, namely

T_{ij_ether} = c1*((u[-i,-k]*u[-j,k])+ (u[-k,-i]*u[k,-j])-1/2*g[-i,-j]* (u [-k,-l]*u[k,l])+ (u[l,-l]*u[-i,-j]) + (u[k]*u[-i,-j,-k])- (u[k,-i,-k]*u[-j])- (u[k,-i]*u[-j,-k])- (u[-i,-k]*u[-j,k])-(u[-i]*u[-j,k,-k])) + c2*(2*(u[-i,-j]*u[k,-k])-1/2*g[-i,-j]*((u[k,-k]))^2 + (g[-i,-j]*u[k]*u[l,-l,-k])+ (g[-i,-j]*u[k,-k]*u[l,-l]) - 2*(u[k,-k,-i]*u[-j])-2* (u[k,-k]*u[-i,-j]))+ c3*((u[k,-i]*u[-j,-k]) + (u[-i,k]*u[-k,-j])-1/2*g[-i,-j]* (u[k,l]*u[-l,-k])+ (u[l,-l]*u[-j,-i])+ (u[k]*u[-j,-i,-k])- (u[k,-i,-k]*u[-j])- (u[k,-i]*u[-j,-k])- (u[-i,-k]*u[-j,k])- (u[-i]*u[-j,k,-k])) + c4*((u[k]*u[l]*u[-i,-k]*u[-j,-l]) + (u[-j]*u[k]*u[l,-i]*u[-l,-k]) + (u[k]*u[-j]*u[l,-k]*u[-l,-i]) - 1/2*g[-i,-j]*(u[k]*u[l]*u[m,-k]*u[-m,-l])+(u[k,-k]*u[-i]*u[l]*u[-j,-l])+(u[k]*u[-i,-k]*u[l]*u[-j,-l]) + (u[k]*u[-i]*u[l,-k]*u[-j,-l])+(u[k]*u[-i]*u[l]*u[-j,-l,-k])-(u[-j,-k]*u[k]*u[l]*u[-i,-l])-(u[-j]*u[k,-k]*u[l]*u[-i,-l])-(u[-j]*u[k]*u[l,-k]*u[-i,-l])-(u[-j]*u[k]*u[l]*u[-i,-l,-k])-(u[-j,-k]*u[-i]*u[l]*u[k,-l])-(u[-j]*u[-i,-k]*u[l]*u[k,-l])-(u[-j]*u[-i]*u[l,-k]*u[k,-l])-(u[-j]*u[-i]*u[l]*u[k,-l,-k]) + lambda*u[-i]*u[-j]

(3)

where λ is the Lagrange multiplier enforcing the unit timelike constraint on the u vector field. At the same time, the u field equation becomes

$$2\, c1\, u^{i;j}{}_{;j} + 2\, c2\, u^{k}{}_{;k}{}^{;i} + 2\, c3\, u^{k;i}{}_{;k} + c4\, (u^{k} u^{l;i} u_{l;k} + u^{k} u^{l}{}_{;k} u_{l}{}^{;i} + 2\, u^{k}{}_{;k} u^{l} u^{i}{}_{;l} + 2\, u^{k} u^{l}{}_{;k} u^{i;l} + 2\, u^{k} u^{l} u^{i}{}_{;kl}) + \lambda\, u^{i} = 0. \qquad (4)$$

The overall goal is to consider static, spherically symmetric solutions to these field equations and we have several reasonable choices of coordinate systems. However, in the weak field case, the isotropic coordinates (t,r,θ,φ) (r is not the usual radial coordinate associated with the areas of two-spheres) with the general line element

$$ds^2 = N(r)\, dt^2 - B(r)\, (dr^2 + r^2\, d\Omega^2) \qquad (5)$$

are natural from the theoretical viewpoint due to the conformal flatness of the metric and are practical because they are most often used in discussions of solar system observations. Computing the connection coefficients and curvature tensors for this metric that appear in the field equations is obviously an enormous task so I used the symbolic math program Maple and its tensor analysis package. With this package one can simply enter the above line element and ask the program to calculate the components of the Einstein tensor. For calculations of the aether stress tensor I gave the u vector field the general spherically symmetric form

$$u = u^{t}\, \partial/\partial t + u^{r}\, \partial/\partial r = a(r)\, \partial/\partial t + b(r)\, \partial/\partial r. \qquad (6)$$

In Maple the time component I designated $a(r)$ the time component and $b(r)$ as the radial piece. With this definition, the computer was able to calculate all the necessary covariant derivatives appearing in the stress tensor. Once this is completed one is left with a set of differential equations for the functions $N(r)$, $B(r)$, $a(r)$, $b(r)$, and $\lambda(r)$. Note that the Lagrange multiplier field is also assumed to be function only of the radial coordinate due to the symmetry of the spacetime. But, we still have the freedom to eliminate the Lagrange multiplier and solve for one of the components of the vector field.

In the Schwarzschild solution it is well known that out of the 4 possible field equations associated with G_{tt}, G_{rr}, $G_{\theta\theta}$,$G_{\varphi\varphi}$, only 2 will be independent, the G_{tt} and G_{rr} , again due to the symmetry of the spacetime. Thus, I selected these components of the metric field equation as two of my master equations along with the 2 equations that fall out of the u field equation. I then solved one of the u equations for lambda in terms of N(r), B(r), a(r), and b(r) and plugged it into the remaining 3 equations. In the final step of the process I used the constraint equation

$$u^i u_i = 1 = N(r)\, a(r)^2 - B(r)\, b(r)^2 = 1 \qquad (7)$$

and solved for a(r) in terms of b(r). The choice of component to eliminate should not be important, but I found that for series expansions choosing b(r) prevented the expansion from choking in Maple. Finally, we are left with 3 very complicated differential equations for 3 unknowns, N(r), B(r), and b(r).

Series Expansions

At this stage we are finally able to address the main question of the behavior of the solution in the asymptotic weak field limit. In this region, far from the source, GR should reduce to Newtonian gravity and the metric should approach flat Minkowski space. In order to examine what happens as r approaches infinity in the GR plus ether model I introduced the change of variables to x = 1/r. Around x = 0 the functions N(x), B(x), b(x) will have power series behavior in the form of

$$N(x) = N(0) + N(1)\, x + N(2)\, x^2$$
$$B(x) = B(0) + B(1)\, x + B(2)\, x^2$$
$$b(x) = b(0) + b(1)\, x + b(2)\, x^2 \qquad (8)$$

The asymptotically flat boundary conditions are N(0) = 1, B(0) = 1. Spherical symmetry seems to say that u field should be rest at infinity (b(0) = 0), but I initially decided to leave it undetermined. In all theories the 1st order terms in x (1/r) should be associated with the Newtonian gravitational potential so what we are really interested in are the corrections to this embodied in the N(2) and B(2) coefficients. Substituting the above forms into the master differential equations and performing a series expansion in Maple around the singular point x = 0 gives a set of algebraic equations that can be solved to produce the local power series solutions. The lowest order solutions immediately show that b(0) = 0 as we expected. In order to simplify my Maple calculations, I went back and fixed b(0) = 0 in the original expansions, a change that will not affect the final results. At the new lowest order (3rd order in x) we find

$$4\,c\_1\,b(1) + 4\,c\_3\,b(1) + 4\,c\_2\,b(1) = 0$$
$$-N(1) - B(1) = 0 \tag{9}$$

so

$$b(1) = 0\,, \quad B(1) = -N(1). \tag{10}$$

At 4th order there are again 2 equations after substituting the previously discovered conditions for b(1) and B(1):

$$-2\,B(2) - c\_4\,N(2) - 3\,c\_3\,N(2) + \frac{3}{4}N(1)^2 - c\_1\,N(2) - \frac{1}{8}c\_1\,N(1)^2 + \frac{1}{8}c\_4\,N(1)^2 + \frac{3}{2}c\_3\,N(1)^2 = 0$$

$$-2\,B(2) - 2\,N(2) + \frac{7}{4}N(1)^2 + \frac{1}{8}c\_4\,N(1)^2 + \frac{1}{8}c\_1\,N(1)^2 = 0 \tag{11}$$

Solving these equations for B(2) and N(2) gives the final result

$$N(2) = -\frac{1}{4}\frac{N(1)^2\,(4 + c\_1 - 6\,c\_3)}{-2 + c\_4 + 3\,c\_3 + c\_1}$$

$$B(2) = \frac{1}{16}N(1)^2\,(-12 + 16\,c\_1 + 18\,c\_3 + 12\,c\_4 + c\_4^2 + 3\,c\_4\,c\_3 + 2\,c\_4\,c\_1 + 3\,c\_1\,c\_3 + c\_1^2)/(-2 + c\_4 + 3\,c\_3 + c\_1)$$

(12)

Some comments on these results are in order. First, there is only one free parameter in the solutions (N(1)) which by analogy with Schwarzschild should be associated with the total mass. This result is not obvious because it was certainly possible that some additional parameter involving an "aether charge" analogous to the electromagnetic charge of Reissner-Nordstrom could have appeared. In fact, examination of eqn. (9), shows that if c1+c2+c3 = 0 then b(1) must remain undetermined and act as the charge-like parameter. This is fits with the analysis of [1], where Jacobson and Mattingly showed that the Reissner-Nordstrom metric is the static spherically symmetric solution in the Einstein-Maxwell theory, which corresponds to c1= -c3, c2= c4= 0. Secondly, in the limit where all the coefficients are zero the solutions do return the results expected from Schwarzschild. In addition, it is pleasing that the lower order result for B(1) matches pure gravity and does not depend on the aether coefficients at all. Finally, the condition b(1) = 0 can be reinterpreted as a(1) = -1/2*N(1), which is an interesting equation relating the time component of the ether field to the mass. It should not be difficult to proceed to higher orders in the expansions, but the N(2) and B(2) equations now provide the first important observational check on the theory.

Post-Newtonian Parameters

In usual analysis of the post-Newtonian corrections to the gravitational field of a static spherical body (in this case, the sun) [4] the Schwarzschild line element is rewritten in terms of isotropic coordinates and those metric coefficients are then expanded to post-Newtonian accuracy. From p.1098 in [4] this takes the following form for a general gravitational theory

$$ds^2 = (1- 2M/r + 2\,\beta(M/r)^2)\, dt^2 - (1 + 2\,\gamma\, M/r)\, [dr^2 + r^2\, d\Omega^2] \qquad (13)$$

where M is the mass of the sun and β and γ are the Eddington-Robertson-Schiff parameters for the theory. γ is defined a measuring the "amount of space curvature produced by a unit rest mass" and β describes the amount of "non-linearity in the superposition law" [4] From the above analysis of the preferred frame model we can simplify read off that (letting N(1) = -2M)

$$\gamma = 1$$

$$\beta = -\tfrac{1}{2}\,(4 + c1 - 6\,c3)/(-2 + c4 + 3\,c3 + c1) \qquad (14)$$

These are clearly not the same parameters as in GR, where both β and γ are 1, or in the popular Brans-Dicke scalar-tensor theory, which has $\beta = 1$ and γ defined in terms of the so called Dicke constant ω as $1+ \omega/2+ \omega$. It is also interesting that the c2 coefficient appears to be unconstrained in this approximation. According to Will [3] an analysis of the perihelion shift of Mercury gives a bound on coefficients in the form $|2\gamma - \beta - 1| < 3 \times 10^{-3}$ implying that $|1- \beta| < 3 \times 10^{-3}$. In comparison, tests of light deflection show that γ-1 has a limit of 3×10^{-4} implying that the coupling constant in Brans-Dicke is very large. Since we can tune the coefficients to be a small as we like, this obviously cannot rule out the model, but it at least it provides the first bound on the aether coefficients. In order to have a comprehensive check on the theory in the solar system we need to consider the full post-Newtonian approximation scheme. This model allows for preferred frame effects due to the motion of the sun and solar system with respect to the frame and is described by ten parameters (2 of which are γ and β). A possible further area of research is to find the complete 10 PPN parameters of the theory. However, any parameters associated with preferred frame effects will involve the velocity relative to this frame. If we assume the velocity the solar system is small, the Eddington-Schiff-Robertson parameters should dominate since the aether terms are already damped by the "c" coefficients. A better test of preferred frame effects may be found in gravitational wave studies [2] and binary pulsar systems.

Black Holes

When analyzing static spherically symmetric black holes we have the option of using Schwarzschild or Eddington-Finkelstein like coordinates. Although Eddington-Finkelstein coordinates have the advantage of regularity at the horizon and the ability to describe interiors, I decided to follow the conventions of [5] and examine the region around an event horizon using the Schwarzschild like coordinates

$$ds^2 = \sigma^2(r)\, N(r)\, dt^2 - 1/\, N(r)\, dr^2 - r^2\, d\Omega^2 \tag{15}$$

Since $N(r_h) = 0$ the differential equations produced by Maple from the field equations have a singular point at r_h. Nevertheless, we can find the local power series solutions around the horizon by making the change of variables $x = r - r_h$ and substituting in the expansions around $x = 0$

$$\begin{aligned} N(x) &= N(0) + N(1)\, x \\ \sigma(x) &= \sigma(0) + \sigma(1)\, x \\ b(x) &= b(0) + b(1)\, x \end{aligned} \tag{16}$$

where $N(0) = 0$. If an event horizon exists we should be able to find solutions to the set of algebraic equations produced by the series expansion around $x = 0$. In the full theory (c1-c4) we find 3 relatively complicated equations that do not have solutions for the remaining parameters. Therefore, I eliminated the terms in the field equation referring to the c4 coefficient and tested the c1-c3 theory. In this case, I found 3 equations that have the following solution

$$\sigma(1) = -\frac{2}{3}\frac{\sigma(0)}{r\_h},\ b(1) = \frac{1}{12}\big(24\, c\_1\, N(1)\, r\_h\, b(0)^2 + 88\, c\_2\, b(0)^4 + 136\, c\_3\, b(0)^4 + 72\, c\_1\, b(0)^4 - 9\, r\_h^{\,2}\, N(1)^2\, c\_1\big) \big/ \big(r\_h\, b(0)\, (4\, c\_3\, b(0)^2 + 3\, r\_h\, N(1)\, c\_1 + 4\, c\_2\, b(0)^2)\big) \tag{17}$$

Note that there are 3 free parameters in contrast to the weak field case, which only involves the mass. N(1) likely refers to the mass of the spacetime while σ(0) and b(0) reflect our ability to tune the parameters of the metric and vector field to get the asymptotic conditions. Although this result strongly suggests that black hole solutions are permitted by the restricted c1-c3 theory there are a number of future areas for research.

Conclusions

Despite the fact that this report described a number of important advances, there are still a many questions to be answered about this aether plus GR model besides the form of the 10 PPN parameters. In the future I plan to determine the status of black holes in the theory definitively by examining whether $T^{ab}{}_{ether}\, \chi^a\, \chi^b$, where χ is the timelike Killing vector, vanishes when evaluated at the horizon as required by the Raychaudhuri equation. Some preliminary investigations suggest that the c4 term will again be problematic while the c1-c3 terms do satisfy the condition. If black holes do exist, what is their nature? The local solutions around the horizon and at infinity should be useful in numerical analysis of the full exterior solution. Similarly, one should be able to examine the interior of the black hole. In situations with matter it is also an interesting to ask about stellar solutions. At the center of the star and at infinity the u field should be at rest so how must aether behave in between in order to satisfy these conditions? Finally, there are the more abstract questions about energy and black hole entropy in theory. The ultimate goal of this research should be to either eliminate this theory as a description of nature or to establish it as valid alternative to GR similar to the Brans-Dicke scalar-tensor theory.

References

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