How Students Use Mathematics in Physics: A Brief Survey of the Literature

Jonathan Tuminaro
University of Maryland
College Park, MD 20742

Introduction

Previous research on the role of mathematics in physics can be classified by the various methods employed by researchers to probe how students use mathematics in physics. Four, sometimes overlapping, approaches have emerged, which will be succinctly categorized in this paper as (i) **observational approach**, (ii) **modeling approach**, (iii) **mathematics knowledge structure**, and (iv) **general knowledge structure**. In the observational approach, researchers observe the mistakes that students make when using mathematics in physics and attempt to explain these observations without explicit reference to the students' knowledge structure or cognitive state. The modeling approach generally starts by observing the difference between experts and novices when using mathematics in physics, and then proceeds by constructing computer models that mimic the respective performances of the two groups. The mathematics knowledge structure program posits theoretical cognitive structures that are specifically implicated when using mathematics. As the name suggests general knowledge structure is similar to mathematics knowledge structure, but the former employs a broader, more general approach. The general knowledge structure program posits the existence of various kinds of cognitive constructs to understand the structure of concepts in general, not restricting the focus to simply concepts in mathematics.

An investigation on the role of mathematics in physics requires an understanding of what it means to "use mathematics in physics". For the sake of this review, the "use of mathematics in physics" will simply mean any time students invoke ideas from mathematics—such as equations, graphs, etc.— to help them understand the physics. This definition is sufficiently broad to span the various levels of detail that have been investigated in the research literature on mathematics learning and physics learning.

The main body of this paper discusses the four approaches employed to understand students' use of mathematics in physics, with a separate section dedicated to each approach. The penultimate section offers a brief discussion about how these different approaches fit together in a coherent whole. The final section proposes future work to be done that will contribute to this field.

Observational Approach

The *observational approach* is a relatively straightforward approach used to probe students' use of mathematics in physics. This approach tacitly assumes that students are rational thinkers who make mistakes when using mathematics in physics because of a small number of inappropriate interpretations. It is the presence of these inappropriate interpretations that explain the reason for student errors when using mathematics in physics.

Every algebraic equation has two main structural features: an equal symbol and variables. From the arrangement of these structures the relationship between the variables can be deduced. So, in order to understand an algebraic equation one must successfully interpret at least three different things: the equal symbol, the variables, and the relationship between the variables. As such, this section is broken up into three subsections that focus on students' misinterpretations of the equal symbol, the nature of a variable in an algebraic equation, and multivariable causation (in the context of thermodynamics).

The equal symbol

As a first attempt to understand students' use of mathematics in physics it is natural to assess their interpretation of what an equation really means. Herscovics and Kieran (1980) and later Kieran (1981) tried to do just that. By examining previous research on a range of students from elementary school to early college students, Kieran concludes that students view the equal symbol as a "do something symbol".

Elementary students when reading arithmetic equations like "3 + 5 = 8" would say "3 and 5 *make* 8". This reading of the arithmetic equation "3 + 5 = 8" was interpreted by Kieran to indicate that the students view the equal symbol as a symbolic prompt to add the first two numbers together. The following example supports this interpretation about how students view the equal symbol. First and second grade students when asked to read expressions like "□ = 3 + 4", would say, "blank equals 3 plus 4", but they would also include that "it's backwards! Am I supposed to read it backwards?" The students read the equations from left to right, like English sentences, in which case the result appears before the two numbers are added together. However, to these students three and four must be added together before a result can be computed.

The previous examples lend credence to the interpretation that elementary school students view the equal symbol as a do something symbol. Kieran argues, however, that this interpretation of the equal symbol is not specific to elementary school students. Kieran cites the following examples, from high school students' written solutions, to argue that these students also see the equal symbol as a do something symbol or an operator symbol:

Solve for *x*: (Byers and Herscovics, 1977)

$$\begin{aligned} 2x + 3 &= 5 + x \\ 2x + 3 - 3 &= 5 + x - 3 \\ 2x &= 5 + x - x - 3 \\ 2x - x &= 5 - 3 \\ x &= 2 \end{aligned}$$

And:

$$\begin{aligned} x + 3 &= 7 \\ &= 7 - 3 \\ &= 4 \end{aligned}$$

Examining these two examples it is seen that both sides of the equations are not always equal. The equal symbol is traditionally used in algebraic equations to indicate a numerical equivalence between two mathematical expressions. That is, the equal symbol separates two mathematical expressions that represent the same numerical value. However, the students do not use the equal symbol in that way in the above examples.

Kieran cites an example from Clement (1980), in which early college students enrolled in a calculus course use the equal symbol as a do something symbol.

(Clement, 1980):

$$
\begin{aligned}
f(x) &= \sqrt{x^2+1} \\
&= (x^2+1)^{1/2} \\
&= \frac{1}{2}(x^2+1)^{-1/2} D_x(x^2+1) \\
&= \frac{1}{2}(x^2+1)^{-1/2}(2x) \\
&= x(x^2+1)^{-1/2} \\
&= \frac{x}{\sqrt{x^2+1}}
\end{aligned}
$$

In this example it's as if the student sees the equal symbol as an arrow that leads to the next step in the problem solution. In the first line the student writes down what the function is. In the third line, which is equal to the first line in the student's solution, the student is calculating the derivative of that function. The student connects these lines in the derivation by an equal symbol, which suggests that the student is using the equal symbol as an arrow or do something symbol and not as an equivalence symbol.

It's not clear from this research whether the interpretation of the equal symbol as a do something symbol is harmful to the students or not. That is, there are no direct instructional implications that can be drawn from this work. Rather, this research only gives insight about how students understand one aspect of equations, namely, the equal symbol.

Variables in Algebraic Equations

Clement, Lochhead, and Monk (1981) videotaped college science students solving simple word problems. The students were instructed to talk aloud throughout the process of solving the problem. The observed students experienced great difficulty in translating the English words from the problem statement into algebraic expressions. Leery that the problem was "simply one of misunderstanding English", Clement *et. al.* developed a set of written questions to further probe this issue. One such question read:

> Write an equation for the following statement: "There are six times as many students as professors at this university." Use *S* for the number of students and *P* for the numbers of professors.

This question was given to 150 calculus-level and 47 non-science major students. The correct answer to this question is $S = 6P$, however 37 percent of the calculus students and

57 percent of the non-science majors answered this question incorrectly, with the most common mistake being $6S = P$.

Clement *et. al.* offered two possible explanations for the students' mistakes. The first explanation, which they called *word order matching*, is direct mapping of the English words into algebraic symbols. So the sentence "there are six times as many students as professors" becomes $6S = P$, simply because that's the order in which the words "six", "student", and "professor" appear in the statement of the problem. However, they offer a second, more interesting explanation for the students' mistakes, which they call *static comparison*. According to this explanation students misinterpreted the very meaning of the variables. The variable S, to students using the static comparison interpretation, does not represent the number of students, but rather is a label or unit associated with the number six. Some students even drew figures like the one below (see Figure 1), which indicates that they recognized that there are more students than professors.

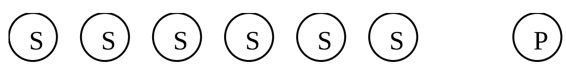

Figure 1. Figure that a student produced to assist in constructing an equation for the following statement: "There are six times as many students as professors at this university."

Multivariable Causation

Research by Rozier and Viennot (1991) shows that, in the context of thermodynamics, some students have trouble parsing the relationship between variables in multivariable problems. Rozier and Viennot analyzed written responses to questions about thermodynamic processes on ideal gases, which could be understood using the equation of state for ideal gases $pV = nRT$. They found that students made two mistakes when interpreting multivariable processes.

First, the students would chunk the variables by mentally reducing the number of variables they would consider in a given process. For example, Rozier and Viennot examined student responses to the following question:

> In an adiabatic compression of an ideal gas, pressure increases. Can you explain why in terms of particles?

The correct response involves the following string of reasoning:

> *volume goes down → number of particles per unit volume goes up* ***and*** *the average velocity of each particle goes up → number of collisions goes up* ***and*** *the average velocity of each particle goes up → the pressure goes up*.

However, a typical student response dropped any consideration about the velocity of the gas particles increasing and would only focus on the number of particles per unit volume increasing. The student response can be represented in the following way:

> *volume goes down → number of particles per unit volume goes up → number of collisions goes up → pressure goes up*.

By only considering the effect that the increase in the number of particles per unit volume had, the students reduced the number of variables that influence this process, and thereby resorted to, what Rozier and Viennot refer to as, *linear reasoning*.

The student's response, in this example, is not necessarily wrong—that is, it doesn't lead to an incorrect conclusion—rather, it demonstrates that students may use a simplified reasoning track to reach the correct conclusion. This example serves only to give insight about the reasoning processes that students use when reasoning about the relationship between variables in multivariable causation.

The second mistake that Rozier and Viennot observed students making when interpreting multivariable causation was the unwarranted incorporation of a chronological interpretation to certain thermodynamic processes. An example of a student response helps bring this point out. When asked to explain why the volume would increase for an ideal gas that is being heated at constant pressure, a student responded:

> The temperature of the gas increases. Knowing that in a perfect gas $pV = nRT$, therefore at constant volume, pressure increases: the piston is free to slide, therefore it moves and volume increases.

In this example the student's response is wrong. It is clear that by allowing the pressure to increase in the solution the student has contradicted the statement of the problem; i.e. that the gas is heated at constant pressure. Rozier and Viennot argue that this contradiction disappears if the stipulation of constant pressure is only temporary, so that the interpretation by the student is understood to progress in time. That is, if the word "therefore" in the student's solution is interpreted to mean "later", the student's solution is no longer contradictory. However, the chronological interpretation present in the student's solution does not come from the equation of state for an ideal gas. The equation $pV = nRT$ represents simultaneous changes in the variables, whereas the student interprets the multivariable causation as being temporal.

Before this section on the observational approach comes to an end, a few words about the implications of the observational approach are in order. The research presented here does not have any direct instructional implications; rather, it serves as a "jumping-off-point" to help us understand how students interpret the different features of an equation. This section focused on student interpretations and student reasoning about equations. The next section will focus on student (and expert) performance while using equations during problem solving.

Modeling Approach

There are two basic components to what I have called the modeling approach. First, one observes the difference between the problem solving skills of the novice and the expert through talk aloud problem solving sessions, or written questionnaires, or both. The second component of the modeling approach is the reason for the name 'modeling approach'. Computer programs are developed with the intent of modeling the performance of either the novice or the expert on similar problem solving tasks.

Larkin, McDermott, Simon, and Simon (1980) articulated four novice/expert differences when solving problems; (i) speed of solution, (ii) backward vs. forward chaining, (iii) uncompiled vs. compiled knowledge, and (iv) syntax vs. semantic interpretations of English statements. The speed of the solution is an obvious difference between novice and expert problem solvers; experts solve problems faster than novices.

A difference that was articulated by Larkin *et. al* is that novices tend to backward chain, whereas experts tend to forward chain when solving problems. This means that novices attack the problem by determining what the end goal is and then work backwards from the end goal toward the initial conditions that are given in the problem statement. In contrast, the expert starts with the initial conditions given in the problem statement and works toward the end goal. This is surprising because backward chaining is generally thought to be a sophisticated problem solving technique.

The third novice/expert difference mentioned above is not a result from direct observations; rather it is a theoretical conjecture about how knowledge is structured for the novice and the expert. Larkin *et. al.* argue that the novice's knowledge must be processed "on the spot" in order to arrive at the problem solution; that is, the novice's knowledge exists in uncompiled form (much like a computer program that is uncompiled). However, the expert may have portions of the problem solution memorized from experience in solving similar problems. Because of these chunks of memorized knowledge, not all of the expert's knowledge must be processed "on the spot" to generate the problem solution; i.e. some of the expert's knowledge exists in compiled form. The difference in the speed of solution for the expert and novice may be accounted for by this difference in knowledge structure.

The fourth novice/expert difference concerns the manner in which English statements are translated into algebraic notation. The novice will tend to write algebraic expressions that correspond with the syntax of the English statements (this is similar to Clement's *word order matching* discussed above). The expert, on the other hand, tends to translate the English statements semantically—that is, in terms of the physics knowledge relevant to the problem—in order to construct algebraic expressions.

Larkin *et. al.* discuss the computer program developed in 1968, called STUDENT, which translates English problem statements into algebraic expressions using the same syntax mapping that is generally associated with students. Larkin *et. al.* use the following problem to discuss how STUDENT works:

> A board was sawed into two pieces. One piece was one-third as long as the whole board. It was exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

To solve the problem STUDENT starts by assigning a variable name (x) to the "length of the board". The first piece mentioned then becomes $x/3$ and the next piece becomes ($x/3 + 4$); therefore, the algebraic expression to be solved is $x = x/3 + (x/3 +4)$.

It was mentioned above that experts use their knowledge of physics to translate English statements into algebraic expressions. The program ISAAC was developed to model this type of expert performance. ISAAC uses schemata to understand ordinary language in terms of idealized levers, fulcrum, ropes, frictionless surfaces, etc.; i.e. it uses its physics knowledge to generate equations from the English statements. For example, ISAAC will recognize a ladder leaning up against a wall as a lever, and associate with that lever the specific properties mentioned in the problem.

What can be concluded from these computer programs? There are programs, like STUDENT, that are capable of mimicking the performance of the novice on certain tasks. There are also programs, like ISAAC, that are capable of mimicking the performance of the expert on certain tasks. Therefore, Larkin *et al* argue that intuition and problem solving "need no longer to be considered mysterious and inexplicable", and with our increased understanding of the expert's knowledge will come new avenues by which to understand the learning processes involved in the acquisition of such knowledge.

Mathematics Knowledge Structure

The previous two sections discussed the observational approach and the modeling approach; this section will discuss the *mathematics knowledge structures* approach. In the mathematics knowledge structure approach, researchers posit various theoretical cognitive structures. A cognitive mechanism, which explains the observed phenomena of the novice and/or the expert using mathematics, can then be constructed from the theoretical cognitive structures. This section will be divided into three subsections entitled *Types of Scientific Knowledge*, *Symbolic Forms*, and *Ontological Structure of Mathematical Entities*. These three subsections will focus on work by Reif and Allen (1992), Sherin (2001), and Sfard (1991), respectively.

Types of Scientific Knowledge

Reif and Allen (1992) developed a cognitive model of "ideally good scientific concept interpretation", which they used to understand the difference between 5 experts and 5 novices solving problems about acceleration. Reif and Allen's model starts by proposing knowledge that falls in three different categories (see Figure 2): (i) *main interpretation knowledge*, (ii) *ancillary knowledge*, and (iii) *form of knowledge*.

Main interpretation knowledge, as the name suggests, is the primary structure implicated in interpreting a scientific concept. Main interpretation knowledge has two major components:

1. **General knowledge**. General knowledge about a scientific concept is divided into three parts.
    a. A precise *definition* is important for any scientific concept and makes up the first part of general knowledge.
    b. *Entailed knowledge* is derivable from the definition, but is not explicitly articulated in the definition.
    c. Lastly, *supplementary knowledge* is related to, but not derivable from the definition.
2. **Case-specific knowledge**. This is knowledge that is applicable in a narrow domain of phenomena. As an example, consider an object moving with constant speed on an oval path. Many students say that the acceleration of the object is directed toward the center of the oval. Although this is true for a circular path, this result is not true for a generic oval path.

The second type of knowledge in Reif and Allen's framework is ancillary knowledge. Like main interpretation knowledge, there are two major components that make up ancillary knowledge. First, interpreting a scientific concept requires one to know when to use their knowledge; i.e. when is it *applicable* [validity] and when is it *useful* [utility]. Second, interpreting a scientific concept requires one to know *how* to use their knowledge; i.e. knowing the rules for applying one's knowledge.

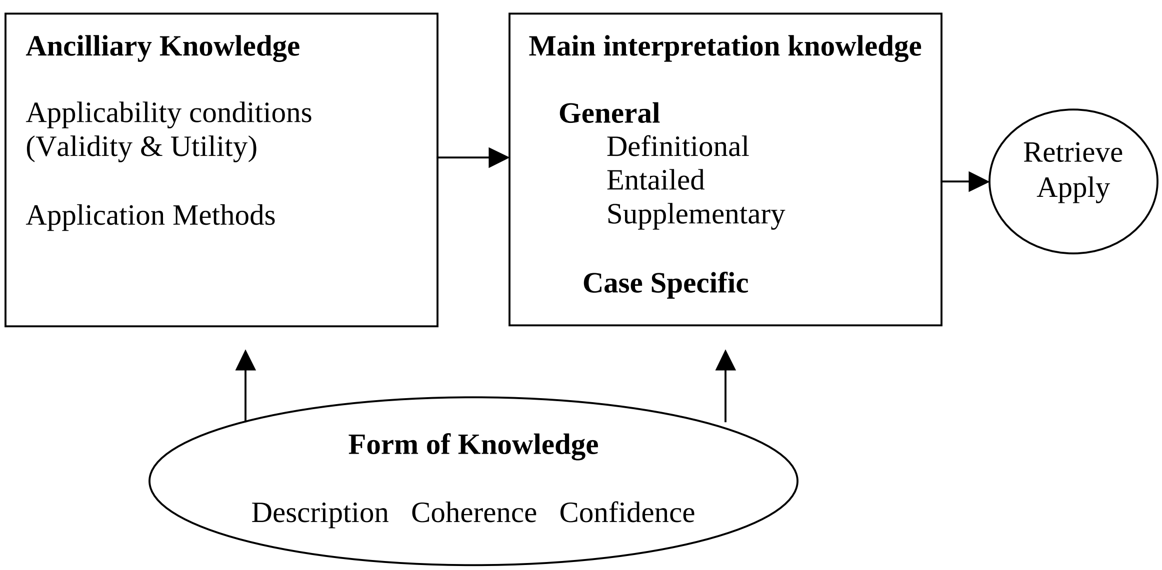

Figure 2. Kinds of knowledge facilitating interpretation of a scientific concept. (Reif and Allen, 1992, p. 10)

The form of knowledge is the third type of knowledge that Reif and Allen proposed, which deals with the organization of the individual's knowledge. The following three components are contained in the form of knowledge:

1. **Description**. An individual's knowledge can have a very precise description or it could be described in vague terms. Either description will effect how the knowledge is applied.
2. **Coherence**. Individual knowledge elements may fit together into a coherent structure or they may be loosely connected fragments.
3. **Confidence**. Confidence in one's knowledge can effect how that knowledge is applied. Over-confidence in one's knowledge may lead to careless mistakes or "incorrect application of the knowledge", whereas under confidence in one's knowledge may prevent the application of appropriate knowledge.

Reif and Allen attempt to categorize the different types of knowledge implicated in the understanding of a scientific concept. The next subsection discusses Sherin's attempt to understand how students use their knowledge to understand physics equations.

Symbolic Forms

Sherin (2001) tried to gain insight as to "how students understand physics equations". He started by collecting data on how students used equations. His data consisted of videotaped sessions in which engineering students solved problems in pairs at a whiteboard. The students in Sherin's study were fairly advanced and did not make structural math errors. These students were enrolled in a third semester physics course at the University of California at Berkeley. From this data Sherin developed a framework, called *symbolic forms*, to interpret how students understand physics equations.

Symbolic forms consist of two parts. The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □... (where the boxes can contain any type of mathematical expression). The *conceptual schema* is a simple structure associated with the form that offers a conceptualization of the knowledge contained in the mathematical expression; this part of the symbolic form is similar to diSessa's *p-prims* (diSessa, 1993).

Examples of the difference between the symbol template and conceptual schema may serve to clarify these definitions (Figure 3). A student would use the symbol template, □ = □, when invoking the conceptual schema of *balancing*. For instance, the utterance, "the normal force of a table on a block is *balancing* the gravitational force of the earth on the block", corresponds with the algebraic expression $N_{T\ on\ B} = W_{E\ on\ B}$, a clear use of the symbol template □ = □. The student also utilizes the same symbol template, □ = □, in association with the conceptual schema *same amount*. For instance, the mathematical expression associated with the utterance, "the velocity of block A is the *same* as the velocity of block B", is $v_A = v_B$; this, again, is a clear use of the symbol template □ = □. To summarize, although the symbol templates were the same for both cases, the conceptual schemata associated with the symbol templates were different; therefore, different symbolic forms are implicated in the two cases.

Utterance	*Conceptual Schema*	*Symbol template*	*Mathematical expression*
"The normal force of a table on a block is balancing the gravitational force of the earth on the block"	Balancing	□ = □	$N_{T\ on\ B} = W_{E\ on\ B}$
"The velocity of block A is the same as the velocity of block B"	Same amount	□ = □	$v_A = v_B$

Figure 3. Different conceptual schema associated with the same symbol template.

Sherin's framework was developed to accommodate algebraic equations for structureless quantities; however, it's unclear whether this same framework would accommodate different types of equations—like vector equations and operator equations—or, if this framework needs to be extended in some way to handle equations that are not simply algebraic equations containing structureless quantities. It may be that different mathematical entities—like vector equations and operator equations—are conceptualized in different ways. The next section discusses two different ways in which mathematical entities are conceptualized.

Ontological Structure of Mathematical Entities

There is no explicit mention of any ontological structure in Sherin's symbolic forms, however Sfard (1991) argues there is an ontological structure to all abstract mathematical notions. According to Sfard, these abstract mathematical notions can be viewed "*structurally*—as objects, and *operationally*—as processes", and that these two views are complementary. For example, a circle can be viewed *structurally* as the locus of all points equidistance from a given point. Or, a circle can be viewed *operationally* as the figure obtained by rotating a compass about a fixed point. Sfard gives various examples of mathematical notions viewed structurally and operationally (these are summarized in Figure 4).

	Operational	*Structural*
Function	Computational process or Well defined method of getting from one system to another (Skemp, 1971)	Set of ordered pairs (Bourbaki, 1934)
Symmetry	[Invariance under] transformation of a geometrical shape	Property of a geometrical shape
Natural number	0 or any number obtained from another natural number by adding one ([the result of] counting)	Property of a set or The class of all sets of the same finite cardinality
Rational number	[the result of] division of integers	Pair of integers (a member of a specially defined set of pairs)
Circle	[a curve obtained by] rotating a compass around a fixed point	The locus of all points equidistant from a given point

Figure 4. Operational and structural descriptions of mathematical notions (Sfard, p5). Note: At some level these maybe formally the same, i.e. to identify a property of a shape one may have to transform the object in their mind—but may not be aware of this mental transformation. That is, the operational and structural interpretations are cognitive not formal differences.

Sfard argues, from a historical point of view, that a structural understanding of a mathematical notion is conceptually more difficult to achieve than an operational understanding. And, the transition from an operational to a structural understanding involves the following three-stage process:

1. *Interiorization*: At this stage, in order for the mathematical notion "to be considered, analyzed and compared it needs no longer to be actually performed" (p. 18).
2. *Condensation*: This phase involves a greater familiarity with the process as a whole, without the need of going through all the details of the process to understand it. That is, "it is like turning a recurrent part of a computer program into an autonomous procedure".
3. *Reification*: This stage is characterized by an ontological shift in how the mathematical notion is viewed, from process to object. This is a sudden and radical shift that offers the "ability to see something familiar in a totally new light."

Sfard summarizes the difference between an operational and structural conception of a mathematical notion along four dimensions (see Figure 5): (1) the general characteristics, (2) the internal representation, (3) its place in concept development, and (4) its role in cognitive processes. Sfard concludes that the operational and structural conceptions of a mathematical entity are complementary and are both useful in problem solving.

	Operational Conception	*Structural Conception*
General Characteristics	Mathematical entity is conceived as a product of a certain process or is identified with the process itself	A mathematical entity is conceived as a static structure as if it was a real object
Internal Representation	Is supported by verbal representations	Is supported by visual imagery
Its place in concept development	Develops at the first stages of concept formation	Evolves from the operational conception
Its role in cognitive processes	Is necessary, but not sufficient, for effective problem-solving learning	Facilitates all the cognitive processes (learning, problem-solving)

Figure 5. Differences between an operational and structural conception of a mathematical notion.

Although, the structural conception comes later than the operational conception of a mathematical notion in Sfard's story, she claims they are two "sides of the same coin". Both conceptions of a mathematical notion are important for understanding and for problem solving.

General Knowledge Structure

The *general knowledge structure* method posits the existence of various kinds of cognitive constructs to understand the structure of concepts in general, not restricting the focus to simply concepts in mathematics. A cognitive mechanism that explains the use of concepts in learning can be constructed from the theoretical cognitive structures.

Two ostensibly distinct frameworks have emerged in the debate about the structure of student knowledge; (i) the *unitary* or *misconceptions* framework (Chi, 1992; Clement, 1983; Carey, 1986) and (ii) the *manifold* or *knowledge-in-pieces* framework (diSessa, 1993). In short, the unitary story of knowledge is that students possess robust cognitive structures, or misconceptions, that need to be torn down, so the correct conception can be erected in its stead. The manifold framework claims that students possess small pieces of knowledge that have developed through everyday reasoning about the world. These small pieces of knowledge can be built upon to foster learning during formal instruction. The remainder of this section discusses two representative theories about concepts that emerge from these different frameworks.

Unitary Knowledge Structure

Chi's (1992) central claim is that concepts exist within ontological categories, and the ontological categories admit an *intrinsic* and a *psychological* reality. The intrinsic reality is "a distinct set of constraints [that] govern the behavior and properties of entities in each ontological category." The psychological reality is "a distinct set of predicates [that] modify concepts in one ontological category versus another, based on sensibility judgment task." So, the intrinsic reality is an objective reality that is imposed by a "sensible" (scientific) community; whereas, the psychological reality is a subjective reality created by the individual. Chi argues that there should be an isomorphism between these two realities in order for learning to occur. Figure 6 shows what an idealized ontology might look like, where an idealized ontology is "based on certain scientific disciplinary standards."

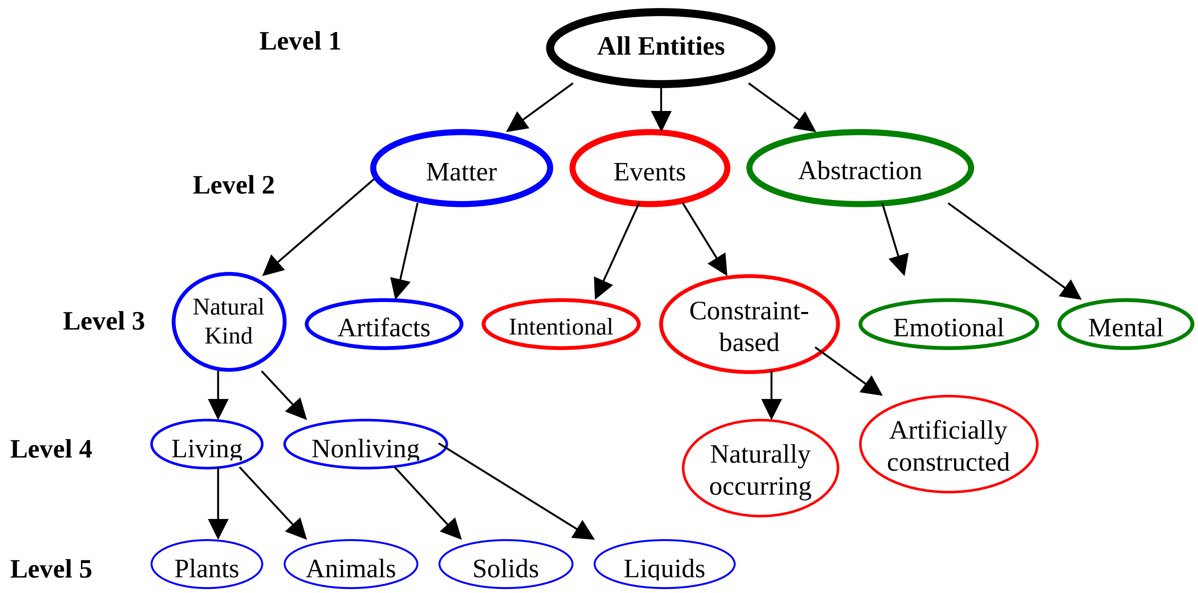

Figure 6. Idealized ontology (Chi, 1992).

To understand conceptual change in Chi's ontological categories model, the details of Figure 6 must be discussed. The six entries along level 3—namely, the ovals entitled Natural Kind, Artifacts, Intentional, Constraint-based, Emotional, and Mental—are six different *branches* or ontological categories. The ontological *tree* refers to the collection of branches or ontological categories that are linked across different levels by arrows (in the figure the ontological tree associated with Matter is in blue). The ontological structure permits two kinds of conceptual change: conceptual change *within* an ontological category, and conceptual change *across* ontological categories. Chi argues that the latter is more difficult and requires different cognitive processes to occur; therefore, it would better be classified as the acquisition of new conceptions rather than conceptual change.

The theory asserts that conceptual change across ontological categories—henceforth called *radical conceptual change*—requires two independent processes. First, the new category must be learned and understood. An example from physics would be the acquisition of the scientific notion of **Force** as a new ontological category. Secondly, radical conceptual change requires the realization that the original assignment of the concept to a particular category is inconsistent with the properties of that category; therefore, the concept must be reassigned to a different category. Staying with the same example from physics, one must realize that the concept of **Impetus**, as articulated by McCloskey (1983), does not belong in the ontological category of **Force**.

The first requirement for radical conceptual change—stated in the previous paragraph—is achieved by learning the new ontological category's properties and learning the meaning of the individual concepts contained within this ontological category. The second requirement for radical conceptual change—reassignment of a concept to a new ontological category—can be achieved in one of three ways.

Firstly, one can "actively abandon the concept's original meaning and replace it with a new meaning." For example, actively realizing that a thrown ball does not posses a quality like **Impetus**, rather the ball simply interacts with other objects via **Forces**.

The second method to reassign a concept to a new ontological category is to allow both meanings of the concept to coexist, in different ontological categories, with either meaning being accessible depending on context. Chi argues that this is probably the most common type of change since many professional "physicists will occasionally revert back and use naive notions to make predictions of everyday events." (It should be noted that some authors see this same example as evidence for knowledge fragments, like *p-prims*, instead of unitary knowledge structures like ontological categories.)

Third, the coherence and strength of the new meaning can be so robust that the replacement of the concept is automatic.

To summarize this subsection, Chi proposes a theoretical framework to understand conceptual change that occurs in learning science. In this framework, concepts exist within a rigid hierarchical structure. In the next subsection the very concept of a scientific concept is brought into question.

Manifold Knowledge Structure

diSessa and Sherin (1998) espouse a theory of concepts that is based on the linkage of fragmented knowledge structures, which they call a *coordination class*. The word coordinate is used in two different senses in the definition of a coordination class. The first is the *integration* of a particular situation into a whole, and the second is the *invariance* of the interpretation across contexts. Along with the two uses of coordination, there are two structural components that make up a coordination class: the *read-out strategies* and the *causal net*. The information that one uses to construct a coordination class is gathered through various *read-out strategies*. Read-out strategies refer to the methods one employs to extract information in various contexts and situations. The *causal net* is the set of implications associated with the coordination class. For example, the existence of a force 'causes' an acceleration, which is essentially captured in Newton's Second Law: $\vec{F} = m\vec{a}$. The meaning of these abstract definitions will be extracted from an example found in the literature.

Wittmann (2002) applies diSessa and Sherin's theory to interpret students' understanding of wave pulses. This work will serve as a concrete example of how the theory of coordination classes may be used by researchers in education research. Wittmann's central claim is that students understand waves as object-like things instead of event-like things. One example that Wittmann discussed involved students' beliefs about pulses traveling on a string. Flicking a taut string with one's hand will generate a wave pulse that travels down the string. The students in Wittmann's study believed the pulse would travel faster if the string were flicked faster. If one were thinking of the wave as being like an object, for example a ball, this would be true. This is consistent with a common p-prim associated with objects, namely *faster means faster*. For example, throwing a ball is accurately described by the *faster means faster* p-prim, since moving one's hand faster when throwing a ball will cause the ball to move faster.

However, in the case of waves—which Wittmann describes as event-like—the *faster means faster* p-prim can be misleading. That is, the *faster means faster* p-prim does apply to the transverse velocity of the wave, which is how the students are using it. So, in this example the p-prim is simply mapped incorrectly onto the physical situation. The speed of the pulse is only dependent on the properties of the media in which it travels, in this case the string. The relative speed at which the hand is moved to generate the pulse has no effect on the relative speed at which the pulse travels down the string.

Wittmann's conclusion is that students coordinate wave around the idea of objects; i.e. the students put waves in the **Object** coordination class, whereas waves really belong in the **Events** coordination class. This coordination, according to Wittmann, occurs along three dimensions. First, the students use their read-out strategies to associate *wave as solid* and *object as point*. Second, the students' motion resources, like *faster means faster* point to wave as object. Third, from examples that are not discussed in this review, the students' interaction resources, like *adding* and *bouncing*, point to wave as object. The motion resources, interaction resources, and read-out strategies all point to *wave as object*. Figure 7 (Wittmann, 2002) summarizes this conclusion.

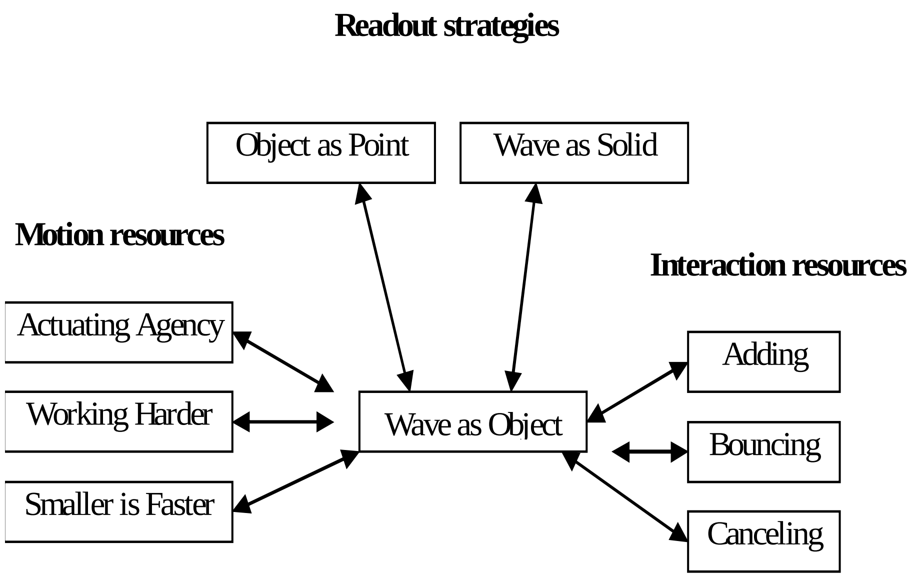

Figure 7. Possible schematic showing reasoning resources that describe an object-like model of waves.

Discussion

The previous four sections look fairly closely at four different attempts to understand how students use mathematics in physics. But how, if at all, do these different approaches fit together? There appears to be a logical flow that leads one approach into the next. The first step to understand how students use mathematics in physics is to systematically observe situations in which students use mathematics or simply document the problems students have when using mathematics in physics. This is the crux of the program in the **observational** approach. The second step in this logical flow—the **modeling** approach—attempts to model the performance or behavior of the students, without reference to the internal cognitive structure that is responsible for the students' performance. The third step—the **knowledge structures** program—attempts to understand the internal cognitive structure that is responsible for the students' performance.

These approaches display a hierarchical structure with increasing levels of sophistication that is reminiscent of the trend in cognitive psychology—from *behavioralism* to *connectionism*—toward a more sophisticated understanding of cognition. (See Anderson, 1995, for a brief review of cognitive psychology.) However, the current stage of understanding about how students use mathematics has not reached the level of sophistication that cognitive psychology has attained. The connectionism movement in cognitive psychology attempts to understand how the cognitive structures interact as a coherent whole to render human cognition, however there is no corresponding program in our understanding of how students use mathematics in physics. This suggests that the next logical step in understanding how students use mathematics in physics is to understand how students' knowledge structures interact to lead students to draw inference about the physics from the mathematics.

References

Anderson, J. (1995). Cognitive Psychology and its Implications. Fourth Ed. W.H. Freeman and Company;
New York, NY.

Byers, V. and Herscovics, N. (1977). Understanding school mathematics. *Mathematics Teaching*; v81 p24-27.

Carey, S. (1986). Cognitive Science and Science Education. *American Psychologist, 41*, 1123-1130.

Chi, M.T.H. (1992). Conceptual change within and across ontological categories: Examples from learning and discovery in science. In R. Giere (**Ed.**) *Cognitive Models of Science, Minnesota Studies in the Philosophy of Science*; v15.

Clement, J., Lochhead, J. and Monk, G.S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly*; v88 n4 p286-290.

diSessa, A.A. (1993). Toward an Epistemology of Physics. *Cognition and Instruction*. v10 n2,3 p105-225.

diSessa, A.A. and Sherin, B.L. (1998). What changes in conceptual change? *International Journal of Science Education*; v20 n10 p1155-1191.

Herscovics, N. and Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher*; v73 n8 p572-580.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*; v12 p317-326.

Larkin, J., McDermott, J., Simon, D.P., Simon, H.A. (1980). Expert and novice performance in solving physics problems. *Science*; v208 p1335-1342.

McCloskey, M. (1983). Naive theories of motion. In D. Gentner & A. Stevens (Eds.), *Mental models* (pp.289-324). Hillsdale, NJ: Erlbaum.

Reif, F. and Allen, S. (1992). Cognition for interpreting scientific concepts. *Cognition and Instruction,* v9 p1-44.

Rozier, S. and Viennot, L. (1991). Students' reasoning in thermodynamics. *International Journal of Science Education*; v13 p159-170.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*; v22, p1-36.

Shama, G. and Redish, E.F., "Students' mental model of motion: Kinematics without vectors," University of Maryland preprint, April, 1999.

Sherin, B. (2001). How Students Understand Physics Equations. *Cognition and Instruction*; v19 n4, p479-541.

Wittmann, M. (2002). The object coordination class applied to wave pulses: analyzing student reasoning in wave physics. *International Journal of Science Education*; v24 n1 p97-118.