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## Born approximation validity conditions

Integral form of time-independent Schrodinger equation for an incoming plane wave in a potential V:

$$\psi(x) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2} \int d^3x' \, \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}')\psi(\mathbf{x}') \tag{1}$$

The (first) Born approximation consists of replacing  $\psi(\mathbf{x}')$  in the integrand by  $e^{i\mathbf{k}\cdot\mathbf{x}'}$ :

$$\psi^{FBA}(x) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{m}{2\pi\hbar^2} \int d^3x' \, \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}') e^{i\mathbf{k}\cdot\mathbf{x}'}.$$
(2)

A sufficient condition for this to be a good approximation is that the difference between  $\psi(\mathbf{x})$  and  $e^{i\mathbf{k}\cdot\mathbf{x}}$  be small in the region  $\mathcal{R}$  where the integral receives important contributions; i.e. if for all  $\mathbf{x} \in \mathcal{R}$ ,

$$\frac{m}{2\pi\hbar^2} \left| \int d^3x' \, \frac{e^{i(k|\mathbf{x}'-\mathbf{x}|+\mathbf{k}\cdot\mathbf{x}')}}{|\mathbf{x}'-\mathbf{x}|} V(\mathbf{x}') \right| \ll 1.$$
(3)

In terms of  $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ , and with  $\mathbf{x}$  chosen as the coordinate origin, this can be written more simply as

$$\frac{m}{2\pi\hbar^2} \left| \int d^3r \, \frac{e^{i(kr+\mathbf{k}\cdot\mathbf{r})}}{r} V(\mathbf{r}) \right| \ll 1. \tag{4}$$

Let's now suppose the potential is negligible outside some radius a, and consider separately the low energy case  $ka \leq 1$  and the high energy case  $ka \gg 1$ .

In the low energy case  $ka \ll 1$ , the exponential doesn't oscillate enough to significantly suppress the integral, so the condition (4) becomes independent of k,

$$\frac{m}{2\pi\hbar^2} \left| \int d^3r \, \frac{1}{r} V(\mathbf{r}) \right| \ll 1. \tag{5}$$

If the potential is characterized by a largest absolute value  $V_{\text{max}}$ , a sufficient condition is thus

$$V_{\max} \ll \frac{\hbar^2}{ma^2}$$
 weakness condition (6)

This can be thought of as saying that the potential energy is much smaller than the kinetic energy associated with the particle being localized in the potential. A similar condition can be derived even if  $V(\mathbf{r})$  has no maximum, as long as the integrand has a maximum. For instance, although  $V(r) = \frac{1}{r}e^{-r/a}$  has no maximum, rV(r) does have one. Note that the weakness condition (6) is sufficient for any ka, since oscillations of the exponential in (4) could only make the integral smaller.

In the high energy case  $ka \gg 1$ , oscillations make the integral smaller, so the Born approximation can be valid even if the potential is not "weak" in the sense of (6). The exponential is unity for **r** opposite to **k**, and is not oscillating rapidly only in the solid angle where  $ka(1 + \cos\theta) \leq 1$ , i.e. where  $\delta\theta^2 \leq 1/ka$ . This means the integral is of order 1/ka times what it is in the long wavelength case  $ka \ll 1$ . Thus a sufficient condition for validity is

$$\frac{V_{\max}}{\hbar^2/ma^2} \ll ka \qquad high \ energy \ condition \tag{7}$$

This is equivalent to the condition  $\int V dt \ll \hbar$  we derived when treating the scattering by first order time-dependent perturbation theory.