HW#9—Phys623—Spring 2016

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1. *Quantum fluctuations of the electromagnetic field*

(a) Calculate the vacuum expectation value ("vev") of the electric field operator, $\langle 0|E^i(\vec{x},t)|0\rangle$.

(b) Calculate the vev of the product of two electric field operators at the same time but different positions, $\langle 0|E^i(\vec{x},t)E^j(\vec{x}',t)|0\rangle$. Isolate the dependence on the separation $|\vec{x}-\vec{x}'|$. Are the fluctuations at two spacelike related points correlated? What happens as the two points approach each other?

(c) Show that the Hamiltonian (16.48) is equivalent to (16.51a).

(d) Taking the vev of the Hamiltonian, find an expression for the energy density of the electromagnetic field in vacuum as an integral over k. If you cut off the upper limit of that integral at some k_c, what is the resulting energy density?

2. This problem concerns spontaneous decay of an excited state of hydrogen with the emission of one photon with momentum $\hbar\mathbf{k}$ and polarization vector ϵ with $\mathbf{k}\cdot\epsilon = 0$. The probability amplitude T for such a transition between atomic states $|i\rangle$ and $|f\rangle$ is proportional to the dimensionless matrix element $\langle f|(M_{orb}+M_{spin})|i\rangle$ with

$$\langle f|M_{orb}|i\rangle = -i(\hbar/mc)\,\epsilon^*\cdot\langle f|e^{-i\mathbf{k}\cdot\mathbf{r}}(\partial/\partial\mathbf{r})|i\rangle \tag{1}$$

$$\langle f|M_{spin}|i\rangle = i(\hbar/2mc)\,(\mathbf{k}\times\epsilon^*)\cdot\langle f|e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\sigma}|i\rangle \tag{2}$$

where $\mathbf{r}$ is the position vector of the electron and $\boldsymbol{\sigma}$ is the vector of Pauli matrices that act on the electron spin.

(a) Show that, for the $2p\to 1s$ transition, $\langle 1s|(M_{orb}+M_{spin})|2p\rangle$ is of order α, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. (This leads to a lifetime of order 10^{-9} s for the $2p$ state.)

The $2s$ state has a much longer lifetime, of order $1/7$ s, because many types of transitions are forbidden. The remaining parts of this problem explore this phenomenon.

(b) Show that $\langle 1s|M_{orb}|2s\rangle = 0$.

(c) Consider the expansion of the spin transition matrix element $\langle f|M_{spin}|i\rangle = \sum_{n=0}^{\infty}(1/n!)\langle f|M^{(n)}_{spin}|i\rangle$, with $\langle f|M^{(n)}_{spin}|i\rangle = i(\hbar/2mc)\,(\mathbf{k}\times\epsilon^*)\cdot\langle f|(-i\mathbf{k}\cdot\mathbf{r})^n\boldsymbol{\sigma}|i\rangle$. Show that $\langle 1s|M^{(n)}_{spin}|2s\rangle$ is zero for $n=0,1$ but non-zero for $n=2$.

(d) Estimate the order of magnitude of $\langle 1s|M^{(2)}_{spin}|2s\rangle$ in powers of α.

(e) Using parts 2a and 2d estimate the lifetime of the 2s state assuming the transition proceeds via the matrix element $\langle 1s|M^{(2)}_{spin}|2s\rangle$.

(f) The actual lifetime (1/7 s) is much shorter than the lifetime you should have obtained in part 2e. Can you think of a reason? (It is *not* that the atom first decays via the allowed $2s_{1/2}\to 2p_{1/2}$ dipole transition. Why?)

3. Strontium is an alkaline earth element. It is thus similar to helium, but with closed inner shells. Four stable nuclear isotopes of strontium exist, of which three have $I = 0$ and one has $I = 9/2$. The ${}^3P_0$ state of a strontium atom is an excited state in which one of the two valence electrons is in a p state. Assume the nuclear spin is zero, so there is no hyperfine interaction. (a) Show that in this case all one-photon transitions from ${}^3P_0$ to the ground state ${}^1S_0$ are forbidden, i.e. at any multipole order and with or without the spin coupling term. (b) Show that there exist allowed one-photon transitions from the ${}^3S_1$ state to the ${}^1S_0$ ground state. Which one dominates? [For the $I = 9/2$ isotope, the hyperfine interaction with the nucleus mixes the ${}^3P_0$ state with states that have allowed one-photon transitions.]