In general: If V^a is a vector density than: $\boxed{D_a V^a = \partial_a V^a}$

i.e. independent of the connection Γ

for a weight W vector density: $D_a V^a = \partial_a V^a + \Gamma^a_{ab} V^b - W\, \Gamma^a_{ab} V^b$.

Alternating ~~...~~ symbol: $\underset{\sim}{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even } 1234 \\ -1 & \text{odd } 1234 \\ 0 & \text{otherwise} \end{cases}$

tensor density of wt. -1

$\tilde{V}^\sigma$ - vector density of wt. $+1$,

$\underset{\sim}{\epsilon}_{\mu\nu\rho\sigma} \tilde{V}^\sigma$ = 3-form (totally antisymmetric tensor of wt. 0), therefor:

$V_3 = \underset{\sim}{\epsilon}_{\mu\nu\rho\sigma} \tilde{V}^\sigma$, $(dV_3)_{\mu\nu\rho\sigma} \Rightarrow \tilde{\epsilon}^{\mu\nu\rho\sigma} (dV_3)_{\mu\nu\rho\sigma} = 3!\, \partial_\mu \tilde{V}^\mu$

because: $\epsilon^{\mu\nu\rho\sigma} \partial_{[\mu} \epsilon_{\nu\rho\sigma]\tau} \tilde{V}^\tau = \epsilon^{\mu\nu\rho\sigma} \partial_\mu \epsilon_{\nu\rho\sigma\tau} \tilde{V}^\tau = \delta^\mu_\tau \partial_\mu \tilde{V}^\tau = 3!\, \partial_\mu \tilde{V}^\mu$.

How this term affects the evolution of the metric?

$$\left\{ h_{ab}, \int \tfrac{1}{2} (\mathcal{L}_N h_{cd}) \pi^{cd} d^3y \right\} = \int \tfrac{1}{2} \mathcal{L}_N h_{cd}(y) \underbrace{\{ h_{ab}, \pi^{cd} \}}_{\delta_a^c \delta_b^d \delta^3(x-y)} d^3y = \tfrac{1}{2} \mathcal{L}_N h_{ab}$$

shift vector

effect of spatial diffeomorph. generated by $\frac{1}{2} N^a$

Similarly:

$$\int N_a \mathcal{H}^a d^3x = \tfrac{1}{2} \int \mathcal{L}_N h_{ab} \pi^{ab} d^3x = -\int \tfrac{1}{2} h_{ab} (\mathcal{L}_N \pi^{ab}) d^3x$$

$$\left\{ \pi_{ab}, -\int \tfrac{1}{2} h_{ab} (\mathcal{L}_N \pi^{ab}) d^3x \right\} = -\tfrac{1}{2} \mathcal{L}_N \pi^{ab}.$$ Diff. for π^{ab}.

$K_{ab} = h_a{}^c \nabla_c n_b = \frac{1}{2} \mathcal{L}_n h_{ab}$.

proof: (symmetry) 1) $n^b K_{ab} = n^b h_a{}^c \nabla_c n_b = \frac{1}{2} h_a{}^c \nabla_c (\overset{-1}{\widehat{n_b n^b}}) = 0$, so

$h_a{}^c \nabla_c n_b = h_a{}^c \underbrace{(h_b{}^d - n_b n^d)}_{\delta_b^d} \nabla_c n_d = h_a{}^c h_b{}^d \nabla_c n_d$. it's spatial.

2) $n_a = f \nabla_a t$, so: $h_a{}^c h_b{}^d \nabla_c (f \nabla_d t) = h_a{}^c h_b{}^d (\nabla_c f) \underbrace{(\nabla_d t)}_{f^{-1} n_d} + h_a{}^c h_b{}^d f (\nabla_c \nabla_d t) =$

$= h_a{}^c h_b{}^d f \nabla_c \nabla_d t$.

So its symmetric in a, b.

$K_{ab} = K_{ba}$