Electromagnetism $(-+++)$:

$$S = \int -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}\, d^4x + \int A_\mu j^\mu \, d^4x \,, \qquad \text{Gauge invariant :} \quad A_\mu \to A_\mu + \partial_\mu \theta$$

$$F_{\mu\nu} \to F_{\mu\nu}$$

in flat space (Minkowsky coords)

$$S = \int -\frac{1}{2} F^{0i} F_{0i} - \underbrace{\frac{1}{4} F^{ij} F_{ji}}_{\frac{1}{2} B^k B_k} \,, \qquad \partial_0 A_0 \text{ never appears} \longrightarrow \boxed{\Pi^0 \equiv 0} \left(\frac{\partial L}{\partial \dot{A}_0} = 0 \right)$$

and $\left(\frac{\partial L}{\partial \dot{A}_i} \right)$: $\boxed{\Pi^i(x) = \frac{\delta L}{\delta \dot{A}_i(x)} = -F^{0i}(x)}$ the electric field.

therefore :

$$H = \int \left\{ \cancel{\Pi^0 \dot{A}_0}^{\,0} + \Pi^i \dot{A}_i + \frac{1}{2} \underbrace{F^{0i} F_{0i}}_{-\Pi^i \Pi_i} + \frac{1}{2} B^2 \right\} d^4x =$$

$$\Pi^i = \partial_0 A_i - \partial_i A_0 \,, \quad \dot{A}_i = \Pi^i + \partial_i A_0$$

In general: $S[A] = \int F(A)\, dx$

$$\frac{\delta S}{\delta A(y)} = \int \frac{\partial F}{\partial A}\bigg|_x \frac{\partial A(x)}{\partial A(y)}\, dx =$$

$$= \int \frac{\partial F}{\partial A}\bigg|_x \delta(x-y)\, dx = \frac{\partial F}{\partial A}\bigg|_y$$

$$= \int \left\{ \frac{1}{2}(\Pi^2 + B^2) + \Pi^i \partial_i A_0 \right\} d^4x = \int \frac{1}{2} \underbrace{(\Pi^2 + B^2)}_{E^2 + B^2} \underbrace{- A_0 \partial_i \Pi^i + A_0 j^0}_{A_0(\partial_i \Pi^i + \rho)} + A_i j^i$$

after imposing the constraints

gauss' law $\Rightarrow$ initial value constraint

See :

$$\partial_t (\nabla \cdot E - 4\pi \rho) = \nabla \cdot \partial_t E - 4\pi \partial_t \rho =$$

$\longleftarrow$ using Maxwell's eqn.

$$= \nabla \cdot (\nabla \times B) - \nabla \cdot (4\pi j) - 4\pi \partial_t \rho =$$

$$= -4\pi (\partial_t \rho + \nabla \cdot j) = 0 \quad \text{if charge is conserved.}$$

Canonical variables: $A_i(x)$, $\Pi^i(x) = -E^i(x)$.