try a new action:

$$S = \frac{m}{2}\int \left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu e^{-1} + e\right) d\lambda \quad , \quad e = e(\lambda)$$ - it's the "tetrad" in 1-d (a 1-d vector ≡ function)

transformation rule:

$$\left.\begin{array}{l} \lambda \longrightarrow \lambda'(\lambda) \quad , \quad d\lambda = \frac{d\lambda}{d\lambda'} d\lambda' \\ e \to e' = \frac{d\lambda'}{d\lambda} e \end{array}\right\} \Longrightarrow \; e\,d\lambda \longrightarrow e' d\lambda'$$

invariant under rep.

Eqn. of motion:

$$\delta_e S = 0 \longrightarrow -\frac{1}{e^2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + 1 = 0 \Longrightarrow e = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

plugging back on the action: $S = \int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda$ what is the original action.

Try to find Hamiltonian form for $S[e(\lambda), x^\mu(\lambda)]$

$$p_e = \frac{\partial L}{\partial \dot{e}} = 0 \quad \longleftarrow \text{constraint.}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m}{e} g_{\mu\nu}\dot{x}^\nu \longrightarrow p_\mu p^\mu = \frac{m^2}{e^2}\dot{x}^\mu \dot{x}_\mu$$

it's not a constraint (depends on arbitrary e and $\dot{x}_\mu$.)

$$H = p\dot{q} - L = p_e\dot{e} + p_\mu \dot{x}^\mu - L = \underbrace{\frac{m}{e} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}_{\frac{e}{m}p^2} - \frac{m}{2}e^{-1}\frac{e^2}{m^2}p^2 - \frac{em}{2} =$$

$$\boxed{H = \frac{e}{2m}(p^2 - m^2)}$$

Hamiltonian action: $S = \int \left\{ p_\mu \dot{x}^\mu - \frac{e}{2m}(p^2 - m^2)\right\} d\lambda$

$\boxed{\delta_e S = 0 \Longrightarrow p^2 - m^2 = 0}$ constraint

e is a lagrange multiplier.

So $H = 0$, but first take the variation, than set to 0.

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = \frac{e}{m}p^\mu = \frac{e}{m}g^{\mu\nu}p_\nu$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = 0$$

↑ if $g_{\mu\nu}$ independent of x^μ, otherwise:

$$= -\partial_\mu\left(\frac{e}{2m}g^{\alpha\beta}(x)\, p_\alpha p_\beta\right) = -\frac{e}{2m}g^{\alpha\beta}{}_{,\mu}\, p_\alpha p_\beta \Longrightarrow$$

equivalent to geodesic eqn for $\dot{x}^\mu$

$$\ddot{x}^\mu = \frac{e}{m}g^{\mu\nu}{}_{,\rho}\dot{x}^\rho p_\nu + \frac{e}{m}g^{\mu\nu}\dot{p}_\nu =$$

$$= g^{\mu\nu}{}_{,\rho}\, g_{\nu\alpha}\dot{x}^\rho\dot{x}^\alpha - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}{}_{,\mu}\, g_{\alpha\gamma}g_{\beta\delta}\dot{x}^\gamma\dot{x}^\delta =$$

$$= \dots = -\dot{x}^\alpha\dot{x}^\beta\Gamma^\mu_{\alpha\beta}$$

, using the fact that $\partial g^{-1} = -g^{-1}\partial g\, g^{-1}$ or $g^{\alpha\beta}{}_{,\gamma} = -g^{\alpha\kappa}g_{\kappa\lambda,\gamma}g^{\lambda\beta}$.