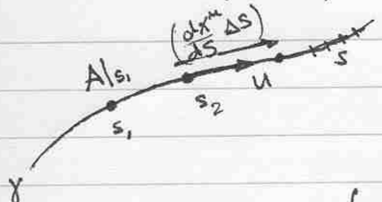
Integration of forms

- Consider a 1-form field : A, defined in M, and consider a curve: γ, on M.

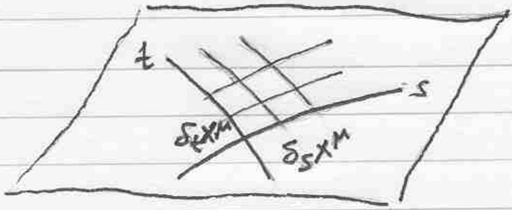


suppose γ is parametrized by s, and has tangent vector $u = \frac{d}{ds}$.

define: $$\int_\gamma A = \lim_{\Delta s \to 0} \sum A\left(\frac{dx^\mu}{ds}(s_i)\Delta s\right) = \int_\gamma A_\mu dx^\mu ,$$

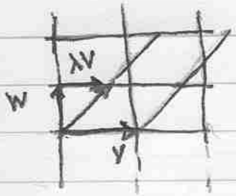
independent of parametrization of the curve.

2-surface : $x^\mu(s,t)$.



$\forall B_{\mu\nu}$, $B_{\mu\nu}\,\delta_s x^\mu \delta_t x^\nu$ is a scalar.

but is it parametrization invariant?



$$B(v, w + \lambda v) = B(v,w) + \lambda B(v,v) \quad , \quad B(v,v) = 0 \ \forall v \implies$$
$$\implies B(v,w) = -B(w,v)$$

Yes, if $B_{\mu\nu} = -B_{\nu\mu}$ (a 2-form)

in other words:

$$\int_S B = \int B_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{dt}\, ds\, dt \quad , \qquad s' = s'(s,t) \ , \ t' = t'(s,t)$$

$$= \int B_{\mu\nu} \left(\frac{\partial x^\mu}{\partial s'}\frac{\partial s'}{\partial s} + \frac{\partial x^\mu}{\partial t'}\frac{\partial t'}{\partial s}\right)\left(\frac{\partial x^\nu}{\partial s'}\frac{\partial s'}{\partial t} + \frac{\partial x^\nu}{\partial t'}\frac{\partial t'}{\partial t}\right) ds\, dt =$$

symmetric   symmetric

$$= \int B_{\mu\nu} \frac{\partial x^\mu}{\partial s'}\frac{\partial x^\nu}{\partial t'} \underbrace{\left(\frac{\partial s'}{\partial s}\frac{\partial t'}{\partial t} - \frac{\partial s'}{\partial t}\frac{\partial t'}{\partial s}\right)}_{\text{Jacobian}} ds\, dt = \int B_{\mu\nu} \frac{\partial x^\mu}{\partial s'}\frac{\partial x^\nu}{\partial t'}\, ds'\, dt' .$$