last time 3/3/03

Algebra of differential forms

$\alpha_p \wedge \beta_q$: $(p+q)$ - form

$\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$ - associative
- distributive

$\mathcal{L}_X$: derivation $\longrightarrow$ $\mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X\alpha\wedge\beta + \alpha\wedge\mathcal{L}_X\beta$

d, i_X : anti-derivation $\longrightarrow$ $+(-1)^p$

$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$
$[\mathcal{L}_X, d] = 0$
$[\mathcal{L}_X, i_Y] = i_{[X,Y]}$
$\{d, i_X\} = \mathcal{L}_X$
$d^2 = 0$
$\{i_X, i_Y\} = 0$

in general: $d\alpha = 0 \longleftrightarrow \alpha$ is closed
$\beta = d\gamma \longleftrightarrow \beta$ is exact
$\rightarrow$ all exact forms are closed.

Hamilton's eqn:

- Phase space Γ and a symplectic 2-form Ω on Γ.
$d\Omega = 0$ (closed)
$X \lrcorner \Omega = 0 \Longrightarrow X = 0$ (non degenerate)
$(X^a \Omega_{ab} = 0)$

Hamiltonian; $H : \Gamma \longrightarrow \mathbb{R}$

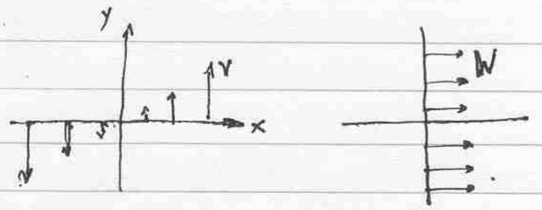
$\boxed{\dot{X}_H \lrcorner \Omega = -dH}$ ← Hamilton's eqn.
$(\dot{X}_H{}^a \Omega_{ab} = -(dH)_b)$

$\dot{X}_H \lrcorner dH = -\dot{X}_H \lrcorner \dot{X}_H \lrcorner d\Omega = 0$ by antisymmetry ($\dot{X}_H$ is unique because Ω is nondegenerated)

$\mathcal{L}_{\dot{X}_H}\Omega = d(\underbrace{\dot{X}_H \lrcorner d\Omega}_{ddH=0}) + \dot{X}_H \lrcorner \underbrace{d\Omega}_{0} = 0$

Difference between directional derivative and Lie derivative

Consider: $V = x\,\partial_y$, $W = \partial_x$, in $\mathbb{R}^2$ (flat) $\rightarrow$



- $V^i \nabla_i W^j = V^i \partial_i W^j = x\partial_y \binom{1}{0} = 0$
- $\mathcal{L}_V W = [V, W] = [x\partial_y, \partial_x] = -\partial_y$

$= -\mathcal{L}_W V \Rightarrow \mathcal{L}_W V = \partial_y$