Hamilton's eq^n and Symplectic transformations.

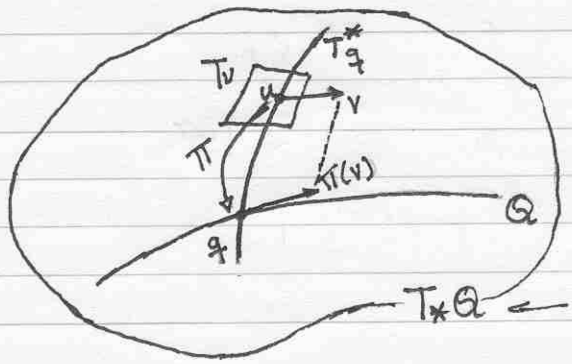
Phase space: $\begin{cases} \text{configuration space manifold}: Q \\ \text{cotangent space at any point } q \in Q : T^*_q \end{cases}$

Why cotangent and not tangent? Because, from $L(q,\dot{q})$, $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and it is a covector

We define the cotangent bundle as the manifold that is locally $Q \times T^*_q$, and has a natural projection $\pi: Q \times T^*_q \longmapsto Q$: suppose we use coordinates q^i on Q and p_i on T^*, we have

$$\pi(q^i, p_i) = q^i,$$ where (q^i, p_i) is a point on the cotangent bundle (T_*Q).

At any point $u \in T_*Q$, we have a tangent space T_u. π define a natural projection from $T_u \longrightarrow T_p$, as:



$T_*Q \leftarrow$ the phase space.

$u = (q^i, p_i)$

$q = q^i$

$$\pi(v^i \partial_{q^i} + v^j \partial_{p_j}) = v^i \partial_{q^i}.$$

→ simplectic form: At the tangent space of $u \in T_*Q$ (or more precisely the cotangent space) we can define a 1-form:

$$\theta = p_i dq^i$$, which action o vectors $v \in T_u$ is: $\theta(v) = p_i \pi(v)^i$

the simplectic form is $\omega = d\theta = dp_i \wedge dq^i$. $d\omega \equiv 0 \leftarrow$ closed.

→ Hamiltonian function: $H: T_*Q \longrightarrow R$, it's defined on T_*Q, $H = H(p^i, q^i)$.

using the symplectic form, we can define the Hamiltonian vector field; X^j

$$\omega_{ij} \frac{dX^j_H}{dt} = \partial_i H \quad \text{or} \quad i_{X_H} \omega = dH$$

explicitelly: $X^j = (q^i, p^i)$ so $\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p^i} \\ \dfrac{dp^i}{dt} = -\dfrac{\partial H}{\partial q^i} \end{cases}$

$i_{\dot{X}} \omega = \omega(\dot{X}) =$

$= (dp_i \wedge dq^i)(\dot{q}^j \partial_{q_j} + \dot{p}_j \partial_{p_j}) =$

$= \dot{q}^i dp_i - \dot{p}_i dq^i = dH =$

$= \dfrac{\partial H}{\partial q^i} dq^i + \dfrac{\partial H}{\partial p^i} dp_i.$