last time:

2/26/03

- metric independent derivatives of Tensors:
  - Lie derivative w.r.t. a vector field: defined in adapted coordinates where $X = \frac{\partial}{\partial s}$ as:

$$\mathcal{L}_X T^{\alpha\ldots}{}_{\beta\ldots} = \frac{\partial}{\partial s} T^{\alpha\ldots}{}_{\beta\ldots}$$

, in general: $\mathcal{L}_X T^{a\ldots}{}_{b\ldots} = X^m \nabla_m T^{a\ldots}{}_{b\ldots} +$

$+ (\nabla_b X^m) T^{a\ldots}{}_{m\ldots} - (\nabla_m X^a) T^{m\ldots}{}_{b\ldots}$

think as a map (transformation) like change in volume of X flow.

  - Exterior derivative (of totally antisymmetric tensors = = differential forms):

ω - p-form

$$(d\omega)_{\mu_0\ldots\mu_p} = (p+1)\,\partial_{[\mu_0}\omega_{\mu_1\ldots\mu_p]}$$

Ex: EM vector potential A

$dA = F$: field strength.

0-form: function, $(df)_\mu = \partial_\mu f$

1-form: vector (covector), $(dA)_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$

2-form: $(dB)_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} = \{\partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}\}$. B is already antisymmetric.

if $F = dA$, $dF = d(dA) \equiv 0 \longleftrightarrow \partial_{[\mu} F_{\nu\lambda]} = 0$

half of Maxwell's eqn $\begin{cases} \nabla\cdot B = 0 \\ \nabla\times E = -\partial_t B \end{cases}$

Commutator of $\mathcal{L}_X$ and d: $[\mathcal{L}_X, d] = 0$

$\mathcal{L}_X$ and $\mathcal{L}_Y$: $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$

inner product: $X \lrcorner\, \omega \equiv i_X\omega = [i_X\omega](x_2,\ldots,x_p) = \omega(X, x_2,\ldots,x_p)$
$= X\cdot\omega = X^a \omega_{a\, a_2\ldots a_p}$

$\mathcal{L}_X = i_X\cdot d + d\cdot i_X \quad \rightarrow$ ex: $\mathcal{L}_X\omega = i_X\cdot d\omega + d\, i_X\omega = X^a \partial_{[a}\omega_{b\ldots c]} + \partial_{[b} X^a \omega_{a\ldots c]}$

$[\mathcal{L}_X, i_Y] = i_{[X,Y]}$: $[\mathcal{L}_X, i_Y]\,\omega = [X,Y]^a \omega_{a\ldots b}$

<u>Algebra of forms</u>

$$(\alpha_p \wedge \beta_q)_{a_1\ldots a_p b_1\ldots b_q} = \alpha_{[a_1\ldots a_p}\beta_{b_1\ldots b_q]}\frac{(p+q)!}{p!\,q!}$$

$\mathcal{L}_X(\alpha\wedge\beta) = (\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta)$ - Leibnitz rule (derivation)

$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta$, $i_X(\alpha\wedge\beta) = (i_X\alpha)\wedge\beta + (-1)^p\alpha\wedge(i_X\beta)$ both are (antiderivations.)