## Curls : Defined for anti-symmetric covariant tensors ( forms)

ex: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$ , is a tensor.

$(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha}) = 0$ if has no torsion and basis vectors commute.

This generilizes :

$$\underbrace{F_{\alpha\beta\gamma\cdots\delta}}_{p} = p!\, \partial_{[\alpha} \underbrace{B_{\beta\gamma\cdots\delta]}}_{p} = (dB)_{\alpha\cdots\delta}$$

facts: $\boxed{d^2 \equiv 0}$

$\boxed{[\mathcal{L}_X, d] = 0}$

$$d^2\omega = \underbrace{\partial_{[\mu}\partial_\nu}_{\text{symmetric in } \mu,\nu.} \omega_{\alpha\dots\beta]} = 0$$

by the definition of Lie derivative: $\mathcal{L}_X \omega_{\alpha\dots\beta} = \frac{\partial x'^\mu}{\partial x^\alpha}\cdots\frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial}{\partial s}\omega'_{\mu\dots\nu}$

in a adapted coordinate system.

Therefore: $d\mathcal{L}_X\omega_{\alpha\beta} = d\left(\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\partial_s\omega'_{\mu\nu}\right) =$

$$= \frac{\partial}{\partial x^{[\gamma}}\left(\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^{\beta]}}\partial_s\omega'_{\mu\nu}\right) = \frac{\partial x'^\rho}{\partial x^\gamma}\frac{\partial}{\partial x'^\rho}\left(\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\partial_s\omega'_{\mu\nu}\right) =$$

$$= \frac{\partial x'^\rho}{\partial x^{[\gamma}}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^{\beta]}}\left(\partial_s\frac{\partial}{\partial x'^\rho}\omega'_{\mu\nu}\right) =$$

$$= \partial_\gamma x'^\rho\,\partial_\alpha x'^\mu\,\partial_\beta x'^\nu\,\partial_s\left(\partial'_{[\rho}\omega'_{\mu\nu]}\right) = \mathcal{L}_X(d\omega).$$

so, $d\mathcal{L}_X = \mathcal{L}_X d \longrightarrow [\mathcal{L}_X, d] = 0$

and

$$\boxed{\mathcal{L}_X\,\cdot = \langle d\cdot, X\rangle + d\langle\cdot, X\rangle}$$

so $\mathcal{L}_X f = \langle df, X\rangle = X^\mu\partial_\mu f$

$\mathcal{L}_X\omega_i = \langle d\omega, X\rangle + d\langle\omega, X\rangle =$

$= X^j\partial_{[j}\omega_{i]} + \partial_i(X^j\omega_j) =$

$= X^j\partial_j\omega_i - X^j\partial_i\omega_j + \partial_i X^j\omega_j + X^j\partial_i\omega_j =$

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$= X^j\partial_j\omega_i + \partial_i X^j\omega_j$