Properties :

- transform as a tensor :

$$(\mathcal{L}_X T)'^{\alpha\dots}{}_{\beta\dots} = \underbrace{\partial_\mu x'^\alpha \cdots \partial_{\beta'} x^\nu}_{\substack{\text{no dependence on } s \\ \text{(and on change of adapted coords)}}} (\mathcal{L}_X T)^{\mu\dots}{}_{\nu\dots} = \frac{\partial}{\partial s}\left(\frac{\partial x'}{\partial x}\cdots\frac{\partial x}{\partial x'}\cdots T^{\dots}{}_{\dots}\right) = \frac{\partial}{\partial s} T'^{\alpha\dots}{}_{\beta\dots} =$$

$$= \mathcal{L}_X T'^{\alpha\dots}{}_{\beta\dots}$$

- satisfies the Leibniz rule : $\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A)\otimes B + A \otimes \mathcal{L}_X B$

- Expressed with covariant derivatives as :

$$\mathcal{L}_X T^{\alpha\dots}{}_{\beta\dots} = \underbrace{X^\gamma \nabla_\gamma T^{\alpha\dots}{}_{\beta\dots}}_{\substack{X^\gamma(\partial_\gamma T^{\alpha\dots}{}_{\beta\dots} + \\ + \Gamma^\alpha_{\gamma\delta} T^{\delta\dots}{}_{\beta\dots} \dots \\ - \Gamma^\delta_{\gamma\beta} T^{\alpha\dots}{}_{\delta\dots} \dots)}} - \underbrace{\nabla_\gamma X^\alpha T^{\gamma\dots}{}_{\beta\dots}}_{(\partial_\gamma X^\alpha + \Gamma^\alpha_{\gamma\delta} X^\delta) T^{\gamma\dots}{}_{\beta\dots}} - \dots + \underbrace{\nabla_\beta X^\gamma T^{\alpha\dots}{}_{\gamma\dots}}_{\dots} + \dots =$$

cancels

$$= X^\gamma \partial_\gamma T^{\alpha\dots}{}_{\beta\dots} - \partial_\gamma X^\alpha T^{\gamma\dots}{}_{\beta\dots} + \partial_\beta X^\gamma T^{\alpha\dots}{}_{\gamma\dots} =$$

$$= \frac{\partial}{\partial s} T^{\alpha\dots}{}_{\beta\dots}$$

in adapted coords
$X^i = 0$ components of X
$X^s = 1$
so : $\partial_\gamma X^\alpha = 0$, $X^\gamma \partial_\gamma = \partial_s$

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ex: $\mathcal{L}_X g_{ab} = X^c \nabla_c g_{ab} + \nabla_a X^c g_{cb} + \nabla_b X^c g_{ac} = \nabla_a X_b + \nabla_b X_a = 2\nabla_{(a} X_{b)}$.
use metric compatible connection

cf. Killing's eqn

$\nabla_{(a}\xi_{b)} = 0 = \mathcal{L}_\xi g_{ab}$.

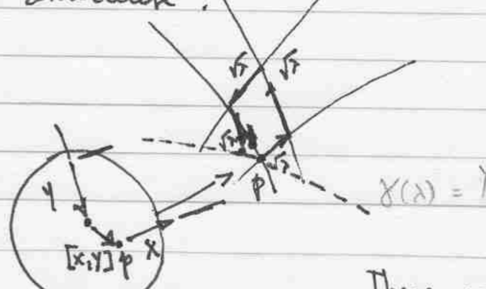
$\mathcal{L}_X Y^\alpha = X^\beta \partial_\beta Y^\alpha - (\partial_\beta X^\alpha) Y^\beta = [X, Y]^\alpha = -[Y, X]^\alpha =$

the commutator of X and Y. $\quad = -\mathcal{L}_Y X^\alpha$

$[X,Y]^\alpha \partial_\alpha f = (X^\beta \partial_\beta Y^\alpha - Y^\beta \partial_\beta X^\alpha)\partial_\alpha f =$
$= X(Y(f)) - Y(X(f))$.

if X and Y are coordinate vector fields , $[X,Y]=0$

but otherwise :



$\gamma(\lambda) = Y_{-\sqrt{\lambda}} X_{-\sqrt{\lambda}} Y_{\sqrt{\lambda}} X_{\sqrt{\lambda}} \big|_p \to [X,Y]\big|_p = \frac{d}{d\lambda}\gamma(\lambda)\big|_0$?

There exist a double adapted coord. system iff $[X,Y]=0$